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On the Woronowicz construction of compact quantum groups. ①

Reference: Woronowicz: Tannaka-Krein duality, Inventions, 1988

Wysoczański: Twisted product structure and... of $U_q(2)$, RMP(2004).

→ two applications of Woronowicz's construction: $SU_q(N)$, $U_q(2)$;
but are there others? This is very hard!

→ and in small dimension? dim 3?

Let $N=3$.

Recall: A C^*-algebra, u fundamental corepresentation, Δ comultiplication, $\varkappa$ coinverse. $\hookrightarrow u=[u_{ke}]_{k,e=1}^{N}$, $u_{ke}\in A$, $\Delta(u_{ke})=\sum' u_{km}\otimes u_{me}$

Th: (Woronowicz). Let $E=(E_{i_1,\dots,i_N})_{i_1,i_2,\dots,i_N=1}^{N}$, $E_{i_1,\dots,i_N}\in\mathbb{C}$, non degenerated such that
$E_{1\cdot},\dots,E_{N\cdot}$ linearly independent
$E_{\cdot 1},\dots,E_{\cdot N}$

and let A be a universal C^*-algebra generated by N^2 elements u_{jk} $(j,k=1,\dots,N)$ satisfying
(u) $u=[u_{ke}]$ is unitary, $\sum u_{ke}^* u_{km}=\delta_{em}1$, $\sum' u_{mk}u_{ek}^*=\delta_{em}1$
(TD) $E_{j_1,\dots,j_N}1=\sum'_{k_1\dots k_N} E_{k_1\dots k_N}u_{j_1k_1}\cdots u_{j_Nk_N}$.
Twisted determinant condition

then (A,u) is a compact quantum group.

Ex: $SU_q(2)$, $q\in\,]0,1[$, or $q\in\mathbb{R}\setminus\{0\}$.
- if $N=2$, $E_{11}=E_{22}=0$, $E_{12}=1$, $E_{21}=q$.
- Then $A=C^*\begin{pmatrix}a&b\\c&d\end{pmatrix}/_{\langle(u),(TD)\rangle}=SU_q(2)$. It can be described by the following relations on α,γ: $\alpha\gamma=q\gamma\alpha$, $\alpha\gamma^*=q\gamma^*\alpha$, $\gamma\gamma^*=\gamma^*\gamma$, $\alpha^*\alpha+\gamma^*\gamma=1$, $\alpha\alpha^*+q^2\gamma\gamma^*=1$

representation: $\alpha e_n=\sqrt{1-q^{2n}}\,e_{n-1}$, $\gamma e_n=q^n e^{i\vartheta}e_n$ for $\vartheta\in[0,2\pi]$.

Eq: $SU_q(N)$. $E_{i_1,\dots,i_N}=\begin{cases}0 & \text{if } i_k=i_l\ (k\neq l)\\ (-q)^{\operatorname{inv}\sigma}, & \sigma=(i_1,\dots,i_N)\in\mathfrak{S}_N\end{cases}$ is nondegenerated

and A is the non-commutative C^*-algebra generated by (u_{jk}): $\left\{\begin{array}{l}u_{ij}u_{ik}=qu_{ik}u_{ij}\ \text{if } j<k\\ \dots\end{array}\right.$

Its irreducible representations are described by Bragiel (1990),
its corepresentations are indexed by Young diagrams.

Last example: $U_q(2)$: $N=3$: $E_{i_1,i_2,i_3}=\begin{cases} 0 & \text{if repetition} \\ (-q)^{3-cc(\sigma)} & , \ \sigma=(i_1,i_2,i_3)\in S_3 \end{cases}$

where $cc(\sigma)$ is the # of cycles in σ.

Its fundamental corepresentation is $u=\underbrace{\begin{pmatrix} a & -qv^*c^* \\ c & v^*a^* \end{pmatrix}}_{w}\oplus v$.

$vv^*=I=v^*v$, $av=va$, $cv=vc$ and (a,c) satisfy $SU_q(2)$ and $\det_q(w)=v^*$

It is represented as $U_q(2)=SU_q(2)\rtimes_\sigma U(1)$. Its corep's are known.

Rem: The same construction yields a commutative example if $N\geq 4$!

Suppose E is a trivial extension of $f: S_n\longrightarrow\mathbb{C}$: $f(\sigma)\neq 0$ if $\sigma\in S_n$; $f(id)=1$.
(Then E is non degenerate)

Q: What kind of QG can we get?

Method: E an array; $E: \mathbb{C}\ni 1\longmapsto \sum E_{ijk}\, e_i\otimes e_j\otimes e_k\in(\mathbb{C}^3)^{\otimes 3}$, (e_i) basis of $\mathbb{C}^3$;
$E^*(e_i\otimes e_j\otimes e_k)=E_{ijk}$

$P=(E^*\otimes I_{(\mathbb{C}^3)^{\otimes 2}})(I\otimes E)$ is the intertwiner of v

We have $P=\begin{pmatrix} c_1 & & \\ & c_2 & \\ & & c_3\end{pmatrix}$, $c_\ell=\sum'_{a,b}\overline{E_{\ell ab}}\,E_{ab\ell}$. If $c_k\neq c_\ell$, then $u_{k\ell}=0$, $u_{\ell k}=0$.

1st case: The three values are different: $u=\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ & u_{22} & u_{23} \\ & & u_{33}\end{pmatrix}=\begin{pmatrix} u_{11} & 0 & 0 \\ 0 & u_{22} & 0 \\ 0 & 0 & u_{33}\end{pmatrix}=u_{11}\oplus u_{22}\oplus u_{33}$

As $c_1\neq c_2$, $u_{12}=u_{21}=0$; By (u), the u_{jj} are unitary; by (TD), they commute and $u_{11}u_{22}u_{33}=1$. This is a description of $A=\mathcal{C}(\mathbb{T}^2)$!

2nd case : $c_1=c_3$, $c_2\neq c_1$: $u=\begin{pmatrix} a & 0 & b \\ 0 & v & 0 \\ c & 0 & d\end{pmatrix}=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\oplus v$

(u) gives $vv^*=I=v^*v$, $aa^*+bb^*=1$, $ac^*+bd^*=0$

(TD) gives $E_{213}1=\sum'_{ijk}E_{ijk}u_{2i}u_{1j}u_{3k}$

$= E_{123}\cdot 0 + E_{132}\cdot 0$

$+ E_{213}\, vad + E_{231}\, vbc$

$+ E_{312}\cdot 0 + E_{321}\cdot 0.$

and thus $v\left(ad+\frac{E_{231}}{E_{213}}bc\right)=1$.

Considering [132], $\left(ad+\frac{E_{312}}{E_{321}}bc\right)v=1$, so that $\boxed{\frac{E_{231}}{E_{213}}=\frac{E_{312}}{E_{321}}}$ or $bc=0$, $ba=0$ so that $c=0$, $b=0$ — this also leads to the 2-dim torus!

We also get $0=E_{112}1=\sum E_{ijk}\dots=E_{132}\,abv+E_{312}\,bav$, so that $ab+\frac{E_{312}}{E_{132}}ba=0$

suppose this $=:\mu$: then $d=a^*v^*$, $av=va$, $b=\mu c^*v^*$, $cv=\frac{\bar\mu}{\mu}vc=e^{-2it}vc$ if $\mu=qe^{it}$

Also, $E_{321}=\bar{\mu}$, $E_{123}=1$, $E_{132}=\alpha$, $E_{213}=\beta$, $E_{231}=\beta\mu$, $E_{312}=\alpha\mu$, $E_{321}=\bar{\mu}$ with $\alpha,\mu\in\mathbb{R}$.

Summary: two families of relations: (R1) $ac+\bar{\mu}ca=0$, $ac^*+\mu c^*a=0$, $cc^*=c^*c$
$a^*a+c^*c=1$, $aa^*+|\mu|^2cc^*=1$

(R2) $vv^*=I=v^*v$, $va=av$, $cv=e^{2it}vc$.

If $|\mu|>1$, note $G_\mu\cong G_{\frac{1}{\bar{\mu}}}$

If μ is real, $\mu\neq 0$: then R1 $\leftrightarrow SU_q(2)$
(R1)+(R2) $\leftrightarrow U_q(2)$.

What if $\mu\in\mathbb{C}\setminus\mathbb{R}$? We would like to have R1 $\sim SU_\mu(2)$.

But such an object does not exist! We also have by R2 that v is not central, so that $U_\mu(2)\neq SU_\mu(2)\otimes_\sigma\mathbb{T}$

Now, the irreducible representations for $U_\mu(2)$ are:

If $|\mu|<1$, 1-dim: $c\mapsto 0$, $a\mapsto z$, $v\mapsto w$ for $z,w\in\mathbb{T}$.

∞-dim: $a\,e_{n,k}=\sqrt{1-|\mu|^{2n}}\,e_{n-1,k}$, $n\in\mathbb{N}$, $k\in\mathbb{Z}$

$c\,e_{n,k}=(-q)^n e^{-(n+2k)it}e_{n,k}$

$v\,e_{n,k}=e_{n,k+1}$

If $|\mu|=1$, 1-dim: $c\mapsto 0$

∞-dim: $a\mapsto 0$, $cf_k=e^{2itk}f_k$, $vf_k=f_{k+1}$

∞-dim: ($\lambda\in\mathbb{C}\setminus\{0,1\}$, $\vartheta\in]0,\pi[$), $af_k=\frac{1}{\sqrt{\lambda}}e^{i\vartheta}f_k$

$cf_k=\frac{1}{\sqrt{1-\lambda^2}}e^{2itk}f_k$

We get the Weyl relations $\approx$ non-commutative torus: $vf_k=f_{k+1}$

$H_0<H$, $cc^*=I=c^*c$, $vv^*=I=v^*v$, $cv=e^{2it}vc$.

What are its corepresentations?

Quantum group structure: $u=\begin{pmatrix} a & 0 & \mu c^*v^* \\ 0 & v & 0 \\ c & 0 & a^*v^* \end{pmatrix}$ gives a C^*-algebra (a,c)
$\Delta(C)\subseteq A\otimes C$
$f_2(a)=|\mu|^2$, $f_2(c)=0$,
$f_2(v)=1$.

We should also discuss the case in which $c_1=c_2=c_3$

$\to$ there is no decomposition: $u_{ij}u_{i\ell}=-\frac{E_{n\ell j}}{E_{nj\ell}}u_{i\ell}u_{ij}$ with $(j,\ell,n)\in S_3$.

$C_j u_{nj}u_{vj}^*+C_\ell u_{n\ell}u_{v\ell}^*=0$ $\to$ either $c_j=0$, $j=1,2,3$. only 4 types of functions $E(\sigma)$: $SU_q(3)$, and two other isomorphic to it

and a very interesting special case: $\bar{u}=[u_{ji}]$ unitary, $u_{ij}u_{rn}^*=u_{rn}^*u_{ij}$,
$u_{ij}u_{ij}^*=0=u_{in}^*u_{ij}$,
$u_{ij}u_{ij}$ is normal and a projection.

But all irreducible reps are 1-dimension. this a commutative subgroup of a reflection group

Then study • $c_j \neq 0$ for one j
and • $c_j \neq 0$ for all j.

Q: The W theorem is just a sufficient condition!
There are QG not of this form