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Intro to the rigidity th. of vN algebras

H sep. H-space, $\mathbb{B}(H) := \{T : H \circlearrowleft \text{bdd linear}\}$ is a Banach $*$-algebra that is unital and in fact a C^*-algebra

A is a <u>vN algebra</u> is "point cv" closed.

Trivial examples: A is a commutative C^*-algebra if $A \simeq C(X)$ for some compact Hausdorff space X; a commutative vN algebra if $A \simeq L^\infty(X,\mu)$ for some probability space (X,μ).

Nontrivial examples: for Γ discrete group, $H = \ell^2(\Gamma)$, $\lambda : \Gamma \to \mathcal{U}(H)$ with $(\lambda_g f)(s) = f(g^{-1}s)$. Then $C^*_\lambda(\Gamma)$, the algebra generated by $\lambda(\Gamma)$ is a C^*-algebra
$L(\Gamma)$ the "pt cv closed" — — vN-

If $\underset{\text{discrete group}}{\Lambda} \overset{\alpha}{\curvearrowright} \underset{\text{standard probability space (countable or } \simeq [0,1])}{(X,\mu)}$ preserving null sets, we also have $\Lambda \overset{\alpha}{\curvearrowright} L^\infty(X)$ and $L^\infty(X) \rtimes \Lambda$ is the vN algebra generated by $L^\infty(X)$ and $\lambda(\Lambda)$ with the relation $\lambda_g f \lambda_g^* = \alpha_g(f)$

Now make the following assumption: Γ is <u>ICC</u> and $\Lambda \overset{\alpha}{\curvearrowright} (X,\mu)$ is <u>free, ergodic, μ-preserving</u>. Then $L\Gamma$ and $L^\infty(X) \rtimes \Lambda$ are II_1 factors.

If M is a vN algebra, M is a factor iff its center is $\mathbb{C}$ (M is <u>simple</u> as a vN algebra)

M is a II_1 factor iff M is a factor different from M_n and finite.
There is a trace $\tau : M \to \mathbb{C}$

Amenability: M factor: M is amenable <u>if</u> M is injective in the sense that for all $A \subset B$ s.t. $A \xrightarrow[\text{u.c.p}]{} M$, there is

$$\begin{tikzcd} B \arrow[dr] & \\ A \arrow[u, phantom, "\cup"] \arrow[r, "\text{c.p}"'] & M \end{tikzcd}$$

Then Γ amenable $\iff$ $L\Gamma$ amenable
and Λ amenable $\iff$ for every action $\Lambda \curvearrowright X$, $L^\infty(X) \rtimes \Lambda$ amenable.

Th (Connes 1976) All amenable factors are classified.

In particular, the amenable II_1 factor is unique: it is, say, R_0.

This means, $L\Gamma$ carries no further information about Γ! neither does ⋊

Q: how about nonamenable factors? This is rigidity theory.

3 central topics:
- calculation of invariants
- structural properties of group factors.
- w^*- super rigidity.

We will focus on the third point: we say Γ, or $\Lambda \curvearrowright X$ is w^*-superrigid if $L\Gamma \simeq L\tilde{\Gamma}$ implies $\Gamma \simeq \tilde{\Gamma}$

Or even, $L^\infty(X) \rtimes \Lambda \simeq L^\infty(Y) \rtimes \tilde{\Lambda}$ implies $\Lambda \simeq \tilde{\Lambda}$, $X \simeq Y$
$\Lambda \curvearrowright X \simeq \tilde{\Lambda} \curvearrowright Y$

This is very far away from amenability.

2^{nd} guy: $\Lambda \curvearrowright X$ case. Suppose $\Lambda \curvearrowright X$ is given and

$R_{\Lambda \curvearrowright X} := \{(x, g\cdot x) \in X \times X : x \in X, g \in \Lambda\}$ Orbit equivalence relation (OER)

and $LR_{\Lambda \curvearrowright X}$ is the OER v.N. algebra.

$$\Lambda \curvearrowright X \xrightarrow{\mathrm{I}} R_{\Lambda \curvearrowright X} \xrightarrow{\mathrm{II}} L^\infty(X) \rtimes \Lambda \text{ vN algebra}$$

this is ergodic theory

this is exactly rigidity theory

if we have w^*-superrigidity $= \mathrm{I} + \mathrm{II}$

Th (Feldman-Moore '77) $R_{\Lambda \curvearrowright X} \simeq R_{\tilde{\Lambda} \curvearrowright Y} \Leftrightarrow L^\infty(X) \rtimes \Lambda \simeq L^\infty(Y) \rtimes \tilde{\Lambda}$
$\cup \qquad \cup$
$L^\infty(X) \simeq L^\infty(Y)$.

Or goal is: If there is $\pi : L^\infty(Y) \rtimes \tilde{\Lambda} \xrightarrow{\sim} L^\infty(X) \rtimes \Lambda$, then
find $\tilde{\pi}$: $L^\infty(Y) \rtimes \tilde{\Lambda} \xrightarrow{\sim} L^\infty(X) \rtimes \Lambda$
$\cup \qquad \cup$
$L^\infty(Y) \xrightarrow{\sim} L^\infty(X)$ and then we have $R_{\tilde{\Lambda} \curvearrowright Y} \simeq R_{\Lambda \curvearrowright X}$.

Popa's intertwining technique:

Th: Popa '01, '03, '04: $M = L^\infty(X) \rtimes \Lambda \supset L^\infty(X) = A$ and $B \subset M$, τ trace on M.

then TFAE: (i) $\exists w_\lambda \in \mathcal{U}(B)$ $\|E_A(b^* w_\lambda a)\|_{2,\tau} \longrightarrow 0$ for all $a, b \in M$.

(ii) $\exists e \in A$ $\exists f \in B$ projections $\exists v \in M$ partial isometry ($v = evf$)

$\exists \vartheta : fBf \hookrightarrow eAe$ $\vartheta(x)v = vx$ for $x \in fBf$.

We write $B \prec_M A$

In (i), $\|x\|_{2,\tau}^2 = \tau(x^* x)$ for $x \in M$ and $E_A(x) = x_e$, where $\overset{M}{\underset{\text{"}}{x}} = \sum_{g \in \Lambda} x_g \lambda_g$

(i) is an estimate on Fourier coefficients — "Fourier expansion"

In (ii), ϑ is a local embedding with an intertwiner v

When
$$
\begin{array}{ccc}
M = L^\infty(X) \rtimes \Lambda & \supset & L^\infty(X) = A \\
\text{\rotatebox{90}{$\cong$}} & & \pi\uparrow \; \pi(L^\infty(Y)) =: B \\
L^\infty(Y) \rtimes \Lambda & \supset & L^\infty(Y)
\end{array}
$$

Under this setting, we can find ϑ as an isomorphism $B \xrightarrow{\sim} A$ and v as a unitary in M [Here A is a so-called Cartan-subalgebra] s.t.

$\underset{v\pi(L^\infty(Y))v^*}{\underset{\text{"}}{vBv^*}} = A = L^\infty(X)$. Then $\tilde{\pi} := \mathrm{Ad}\, v \circ \pi$ works!

Our goal becomes: to get $\pi(L^\infty(Y)) \prec_M L^\infty(X)$

Ozawa-Popa '07: If $\mathbb{F}_n \curvearrowright (X, \mu)$ is a profinite action, then (II) holds.

Popa-Vaes '09: There is a $\Lambda \curvearrowright X$ w^*-superrigid.

$\Lambda := PSL(n, \mathbb{Z}) \underset{T_n}{*} PSL(n, \mathbb{Z})$ with $T_n \subseteq PSL(n, \mathbb{Z})$, $T_n = \{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \}$, for some n and some $\Lambda \curvearrowright X$

Group case: Th Ioana-Popa-Vaes '10: there is Γ w^*-superrigid,

$\Gamma := \mathbb{Z}/n\mathbb{Z} \wr_I \Lambda$ for some $\Lambda \curvearrowright I$ and some n

Idea: reduce this case to the previous one.