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Cumulants associated to n.c. independence

There are several n.c. independences; free.
One associate cumulants to them, cf. Voiculescu, Speicher.
Monotone independence is more difficult and has not been defined before.

1 Independences on a n.c. probability space $(\underset{*\text{-algebra}}{\mathcal{A}}, \underset{\text{state}}{\varphi})$

① tensor independence: let $\{\mathcal{A}_i\}_{i=1}^{\infty}$ be a seq of $*$-subalgebras of $\mathcal{A}$. It is tensor independent if $\varphi(a_1 \dots a_n) = \prod_{i=1}^{\infty} \varphi\Big(\overrightarrow{\prod_{\substack{j=1 \\ a_j \in \mathcal{A}_i}}^{n}} a_j\Big)$ "$\rightarrow$" means that order should be preserved.

where $\varphi(\emptyset) = 1$.

② free independence (or freeness): here $\mathcal{A}$ should be unital and $\mathcal{A}_i \ni 1_{\mathcal{A}}$. It is free [independent]

if $\varphi(a_j) = 0$ and $a_j \in \mathcal{A}_{i_j}$ with $i_1 \neq i_2 \neq \dots \neq i_n$ implies $\varphi(a_1 \dots a_n) = 0$

Application: If $\{\mathcal{A}_1, \mathcal{A}_2\}$ are free, then $\varphi(ab) = \varphi(a)\varphi(b)$ for $a \in \mathcal{A}_1$, $b \in \mathcal{A}_2$
In fact, let $\mathring{a} = a - \varphi(a)1$, $\mathring{b} = b - \varphi(b)1$, so that $\varphi(\mathring{a}) = \varphi(\mathring{b}) = 0$ and
$\varphi(\mathring{a}\mathring{b}) = 0$, i.e., $\varphi(ab) - \varphi(a)\varphi(b) - \cancel{\varphi(a)\varphi(b)} + \cancel{\varphi(a)\varphi(b)} = 0$

In fact, you can calculate any mixed moments. When computations get involved, cumulants help!

③ Boolean independence: here $\mathcal{A}_i \not\ni 1_{\mathcal{A}}$ (in general). It is Boolean independent if

$\varphi(a_1 \dots a_n) = \prod_{j=1}^{n} \varphi(a_j)$ as soon as $a_j \in \mathcal{A}_{i_j}$ (in general (otherwise it is trivial)), $i_1 \neq i_2 \neq i_3 \neq \dots \neq i_n$.

④ Monotone independence: here $\mathcal{A}_i \not\ni 1_{\mathcal{A}}$. It is monotone independent if

$\varphi(a_1 \dots a_n) = \varphi(a_j)\varphi(a_1 \dots a_{j-1} a_{j+1} \dots a_n)$ if $a_i \in \mathcal{A}_{k_i}$ and
$k_{j-1} < k_j > k_{j+1}$ $(2 \le j \le n-1)$
or $k_1 > k_2$ $(j=1)$
or $k_{n-1} < k_n$ $(j=n)$

These independences appeared in some models.
They are basic in the sense that there are no other ones! [the crucial condition is associativity of independence]

<u>Remark</u>: Independence may be considered as a "computation formula for mixed moments"

Associativity is:

Given $\{\mathcal{B}, \mathcal{C}\}$ independent and $\{*\text{-alg}\{\mathcal{B}, \mathcal{C}\}, \mathcal{D}\}$ independent

What about $\{\mathcal{B}, *\text{-alg}\{\mathcal{C}, \mathcal{D}\}\}$, $\{\mathcal{C}, \mathcal{D}\}$? They are also independent, and conversely!

2 Partition of a set and independence.

— Combinatorics of partitions.

Given $\{A_i\}$, how are the elements $a_1 \dots a_n$ positioned? I.e., to which A_i do they belong?

Example: consider $a_1 a_2 a_3 a_4 a_5$, where $a_1, a_3 \in A_1$, $a_2 \in A_2$, $a_4, a_5 \in A_4$: $a_1 a_2 a_3\ a_4 a_5$

This may be described as a partition $\{V_1, V_2, V_3\}$, $V_1 = \{1,3\}$, $V_2 = \{2\}$, $V_3 = \{4,5\}$

If we know the position structure, we can compute the expectation / independence

In the case of monotone independence,
$$\varphi(a_1 \dots a_5) = \varphi(a_1 a_2 a_3)\varphi(a_4 a_5)$$
$$= \varphi(a_2)\varphi(a_1 a_3)\varphi(a_4 a_5)$$
$$= \prod_{h=1}^{3} \varphi\Big(\prod_{i \in V_h}^{\to} a_i\Big)$$

the result follows the partitions.

But sometimes the result is much "smaller" than the given partition:

Suppose $a_1, a_3 \in A_3$, $a_2 \in A_2$, $a_4, a_5 \in A_1$: then
$$\varphi(a_1 \dots a_5) = \varphi(a_1)\varphi(a_2 a_3 a_4 a_5)$$
$$= \varphi(a_1)\varphi(a_3)\varphi(a_2)\varphi(a_4 a_5)$$

Here the result does not follow the given partitions!

Here the necessary and sufficient conditions for the expectation "not to be separated":

1. in the tensor case: no condition at all
2. in the free case: $a_1 \dots a_n \sim NC(n)$ [non crossing partition]
3. in the Boolean case: $a_1 \dots a_n \sim I(n)$ [interval partition: no block covers another one!]
4. in the monotone case: $a_1 \dots a_n \sim \mathcal{M}(n)$, the monotone partition! a special structure is required! → indices.

3 Cumulants (one variable case)

Def.: $\{K_n(a)\}_{n=1}^{\infty}$ is called cumulants if

1. a, b independent $\Rightarrow K_n(a+b) = K_n(a) + K_n(b)$.
2. $K_n(\lambda a) = \lambda^n K_n(a)$ for $\lambda \in \mathbb{C}$
3. $\exists P_n(x_1, \dots, x_{n-1})$ $K_n(a) = \varphi(a^n) + P_n(K_1(a), \dots, K_{n-1}(a))$

Idea: $\varphi((a+b)^n)$ is difficult to compute! Especially in the free case. Cumulants help.
In classical probability, these are the coefficients of the log of the Fourier transform.
Fourier tr. behave multiplicatively, log makes it additive.

Voiculescu's def^n is based on Toeplitz operators. I don't understand well V's def.
For each a, define $N.a := a^{(1)} + \dots + a^{(N)}$ where the $a^{(i)}$ are copies of a, i.e., $\varphi((a^{(i)})^n) = \varphi(a^n)$, that are independent in the considered sense in the corresponding extended space: take the free product of the probability spaces.

<u>Lemma</u>: $\varphi((N.a)^n)$ is a polynomial in N

Example: $(n=1)$. $\varphi(N.a) = \sum_{i=1}^{N} \varphi(a^{(i)}) = N\varphi(a)$.

$(n=2)$. $\varphi((N.a)^2) = \sum_{i=1}^{N} \varphi((a^{(i)})^2) + \sum_{i \neq j} \underbrace{\varphi(a^{(i)} a^{(j)})}_{\varphi(a^{(i)})\varphi(a^{(j)}) \text{ for any independence}}$

$= N\varphi(a^2) + \frac{1}{2}N(N+1)\underbrace{\varphi(a)^2}_{\varphi(a)^2}$, that is, a polynomial in N

for $n \geq 3$, the result is independent of the type of independence.

The proof is done by induction on n.

<u>Def</u>: $K_n(a) =$ " coefficient of N^1 in $\varphi((N.a)^n)$ " (i.e., the linear term)
→ thus linearity of K_n is expected.

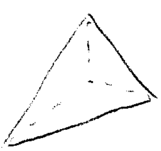
<u>Theorem</u>: ① For tensor cumulants K_n, $\varphi(a^n) = \sum_{\pi \in \underbrace{P(n)}_{\text{all partitions}}} K_\pi(a)$ where $K_\pi(a) = \prod_{V \in \pi} K_{|V|}(a)$.

② For free cumulants K_n, $\varphi(a^n) = \sum_{\pi \in NC(n)} K_\pi(a)$

③ For Boolean cumulants K_n, $\varphi(a^n) = \sum_{\pi \in I(n)} K_\pi(a)$

④ For monotone cumulants, $\varphi(a^n) = \sum_{\pi \in \mathcal{M}(n)} \frac{1}{|\pi|!} K_\pi(a)$

→ we have to think of some additional structure on $\mathcal{M}(n)$, which is very large!



Given $\{V_1, V_2\}$, (V_1, V_2) and (V_2, V_1) are distinguished ...
→ Set of ordered partitions = faces of a Cayley graph (of the symmetric group).