Recent advances in differentiability in metric measure spaces

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Zero order calculus

(X, d, μ) metric measure space, μ Borel regular measure

Coifmann-Weiss 70′ Spaces of homogeneous type.

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Definition

μ is doubling if $\exists\, C > 0$ constant such that

$$0 < \mu(B(x, 2r)) \leq C\, \mu(B(x, r)) < \infty \quad \forall\, x \in X, r > 0.$$

X complete + μ doubling $\Longrightarrow$ X proper

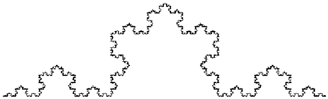
- Lebesgue points
- Vitali Coverings
- Maximal operator...

Examples

- $(\mathbb{R}^n, |\cdot|, \mathscr{L}^n)$ $C = 2^n$
- $(C, |\cdot|, \mathscr{H}^{\frac{log2}{log3}})$

- $([0,1], |x-y|^{1/2}, \mathscr{H}^2)$

$$f(x) = x \qquad \frac{|f(x) - f(y)|}{|x-y|^{1/2}} = |x-y|^{1/2} \xrightarrow{y \to x} 0$$



A curve in X is a continuous mapping $\gamma : [a,b] \to X$.
A rectifiable curve is a curve with finite length.

First order analysis

Rademacher Theorem Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a Lipschitz function. Then f is differentiable $\boxed{\mathscr{L}^n\text{-a.e.}}$ $x \in \mathbb{R}^n$.

First order analysis

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- $\mu << \mathscr{L}^n$ OK
- $\mu = \delta_{x_0} \rightsquigarrow (\mathbb{R}, |\cdot|, \mu)$
- $\mu =$ length of a (lipschitz) curve $\rightsquigarrow (\mathbb{R}^2, |\cdot|, \mu)$

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Stepanov Theorem $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable $\mathscr{L}^n$-a.e. in $S(f)$,

$$S(f) := \left\{ x \in \mathbb{R}^n : \operatorname{Lip} f(x) := \limsup_{\substack{y \to x \\ y \neq x}} \frac{|f(x) - f(y)|}{|x - y|} < \infty \right\}.$$

First order analysis on metric spaces

CHEEGER DIFFERENTIABLE STRUCTURE
Cheeger 99

p-POINCARÉ INEQUALITIES 99
Heinonen - Koskela
Hajłasz, Shanmugalingam..

- NON-LINEAR POTENTIAL THEORY
- GRAPHS
- NON-EMBEDDABILITY
- Q-C MAPS
- SUB-RIEMANNIAN GEOMETRY ...

Lip-lip CONDITION
Keith 03

ALBERTI REPRESENTATIONS
Alberti 93, A-Csörnyei-Preiss ?
Bate 12

STRUCTURE OF NULL SETS

DERIVATIONS
Weaver 99 Gong 12

METRIC CURRENTS

MMS WITH RICCI CURVATURE BOUNDED BELOW
Ambrosio-Gigli-Savaré 08
Lott-Villani 09, Ohta, Sturm

OPTIMAL TRANSPORT
ALEXANDROV SPACES
GRADIENT FLOWS

ANALYSIS ON FRACTALS
Kigami 01, Kusuoka, Goldstein
Barlow-Bass ..

DIRICHLET FORMS
BROWNIAN MOTION

Lipschitz function spaces

(X, d) metric space

Definition

A function $f : X \longrightarrow \mathbb{R}$ is Lipschitz if there is a constant $C > 0$ such that

$$|f(x) - f(y)| \leq C\, d(x, y) \quad \forall x, y \in X.$$

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- $\star$ $\mathrm{LIP}(X) = \{f : X \longrightarrow \mathbb{R} : f \text{ is Lipschitz}\}$
- $\star$ $\mathrm{LIP}^\infty(X) = \{f : X \longrightarrow \mathbb{R} : f \text{ is Lipschitz and bounded}\}$

$$\|f\|_{\mathrm{LIP}^\infty} = \|f\|_\infty + \mathrm{LIP}(f)$$

Pointwise Lipschitz function spaces

Definition

Given a function $f : X \to \mathbb{R}$ the pointwise Lipschitz constant of f at $x \in X$ is defined as

$$\operatorname{Lip} f(x) = \limsup_{\substack{y \to x \\ y \neq x}} \frac{|f(x) - f(y)|}{d(x,y)}.$$

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Example

If $f \in C^1(\Omega)$, $\Omega \overset{\text{op}}{\subset} \mathbb{R}^n$ (or of a Riemannian manifold), then

$$\operatorname{Lip} f(x) = |\nabla f(x)| \quad \forall x \in \Omega.$$

Classical Poincaré inequality

One way to view the Fundamental Theorem of Calculus is:

$$\boxed{\text{infinitesimal data} \rightsquigarrow \text{local control}}$$

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This principle can apply in very general situation in the form of a Poincaré inequality:

$$\exists\, C = C(n) > 0 \colon \forall B \equiv B(x,r) \subset \mathbb{R}^n \; \forall f \in W^{1,p}(\mathbb{R}^n) (1 \le p < \infty)$$

$$\fint_B |f - f_B| \, d\mathscr{L}^n \le C(n)\, r \left(\fint_B |\nabla f|^p \, d\mathscr{L}^n \right)^{1/p}$$

Notation:

$$\fint_B f \, d\mathscr{L}^n = f_B = \frac{1}{\mathscr{L}^n(B)} \int_B f \, d\mathscr{L}^n$$

Applications: Harmonic Analysis and PDEs

Poincaré inequalities in metric measure spaces

(X, d, μ) metric measure space

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Poincaré inequalities in metric measure spaces

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Proof.

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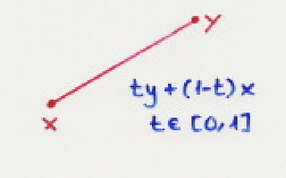
$$\text{FTC} : f(x) - f(y) = \int_0^1 \langle \nabla f(ty + (1-t)x), (y - x) \rangle dt$$



Poincaré inequalities in metric measure spaces

(X, d, μ) metric measure space

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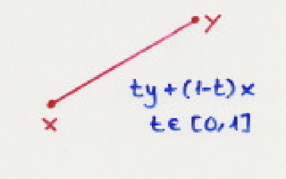

$$|f(x) - f(y)| \leq \int_{[x,y]} |\nabla f|$$

$\ldots$

Poincaré inequalities in metric measure spaces

(X, d, μ) metric measure space

$$\fint_B |f - f_{|B}| d\mathscr{L}^n \leq C(n)\, r \Big(\fint_B \underbrace{|\nabla f|^p}_{?} d\mathscr{L}^n \Big)^{1/p}$$

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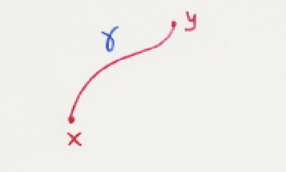


$$|f(x) - f(y)| \leq \int_\gamma |\nabla f|$$

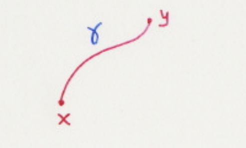
$\ldots$

Poincaré inequalities in metric measure spaces

Definition
Heinonen-Koskela 98

A non-negative Borel function g on X is an upper gradient for $f : X \to \mathbb{R} \cup \{\pm\infty\}$ if

$$|f(x) - f(y)| \leq \int_\gamma g,$$



$\forall x, y \in X$ and every rectifiable curve γ_{xy}.

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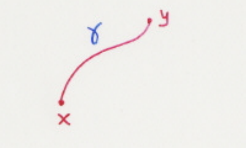

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Examples

- If there are no rectifiable curves in X then $g \equiv 0$ is an upper gradient of every function.
- If $f \in \mathrm{LIP}(X)$ then $g \equiv \mathrm{LIP}(f)$ and $g(x) = \mathrm{Lip}\, f(x)$ are upper gradients for f.

Poincaré inequalities in metric measure spaces

Definition

Heinonen-Koskela 98

Let $1 \leq p < \infty$. (X, d, μ) supports a weak p-Poincaré inequality if there exist constants $C_p > 0$ and $\lambda \geq 1$ such that for every function $f : X \to \mathbb{R}$ and every upper gradient g of f, the pair (f, g) satisfies

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq C_p \, r \left(\fint_{B(x,\lambda r)} g^p d\mu \right)^{1/p}$$

$\forall \, B(x, r) \subset X$.

<u>Notation</u>:

$$\fint_B f \, d\mu = f_B = \frac{1}{\mu(B)} \int_B f \, d\mu$$

Examples

- $(\mathbb{R}^n, |\cdot|, \mathscr{L}^n)$
- Riemannian manifolds with non-negative Ricci curvature
- Heisenberg group with its Carnot-Carathéodory metric and Haar measure $\rightsquigarrow$ Subriemannian geometry
- Boundaries of certain hyperbolic buildings: Bourdon-Pajot spaces $\rightsquigarrow$ Geometric group theory
- Laakso spaces, …

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Cheeger 99 Keith 04

$$\left.\begin{array}{r} X \text{ complete and } p\text{-PI} \\ \mu \text{ doubling} \end{array}\right\} \Longrightarrow X \text{ admits a “differentiable structure”}$$

Geometric implications of p-Poincaré inequalities

- X is connected

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Definition

A metric space (X, d) is quasiconvex if there exists a constant $C \geq 1$ such that for each pair of points $x, y \in X$, there exists a curve γ connecting x and y with

$$\ell(\gamma) \leq Cd(x, y).$$

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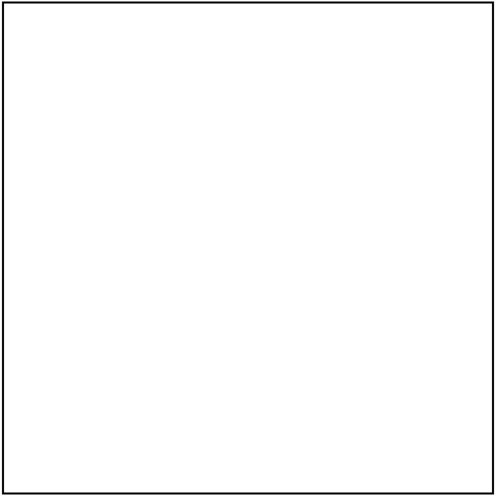
- Heinonen-Koskela 98, Kinnunen-Latvala 02, Saloff-Coste 02, Keith 03, Miranda 03, Korte 07, ...

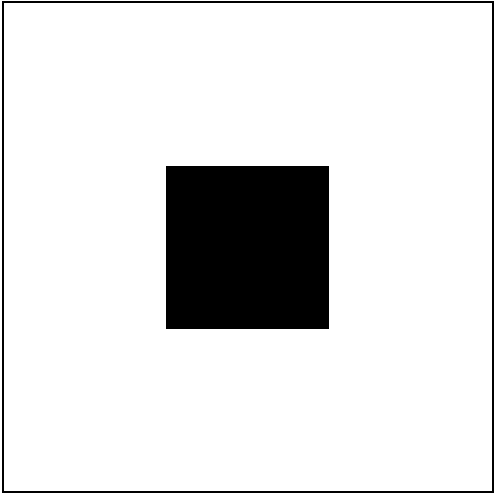
Sierpiński carpet

$$Q_0 = [0,1]^2$$



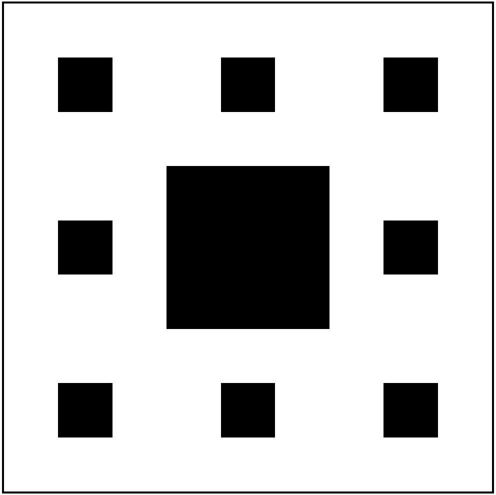
A counterexample: Sierpiński carpet

Q_1



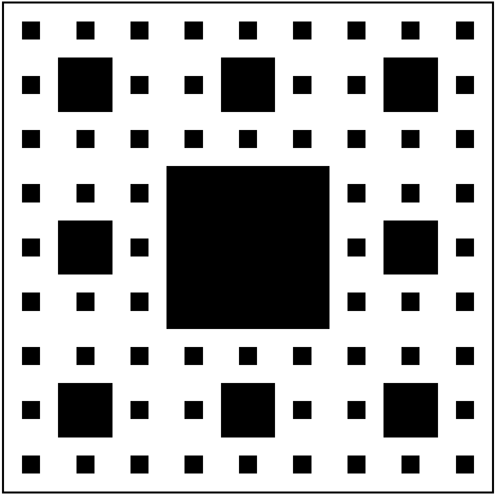
Sierpiński carpet

Q_2



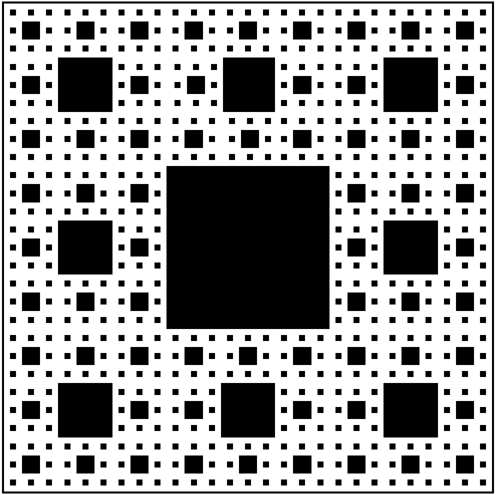
Sierpiński carpet

Q_3



Sierpiński carpet

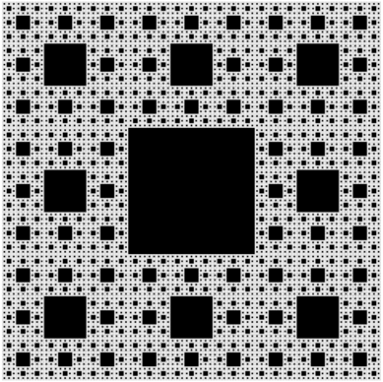
Q_4



Sierpiński carpet

Sierpiński carpet: $S_3 = (X, d, \mu)$

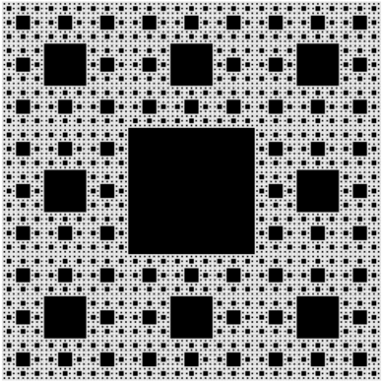
$$d = d_{e|X}$$



Equally distributing unit mass over Q_n leads to a natural probability doubling measure μ on S_3.
(μ is comparable to $\mathcal{H}^s, s = \frac{\log 8}{\log 3}$).

Counterexample

- (S_3, d, μ) does not admit a 1-PI



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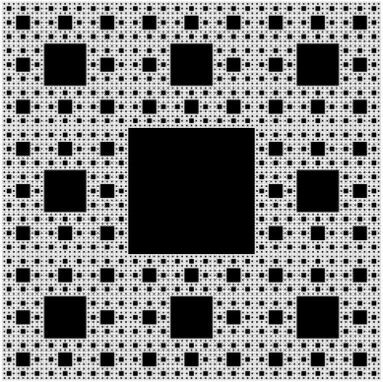
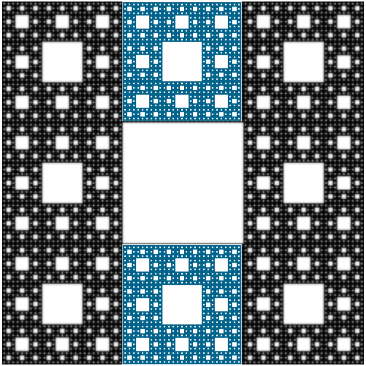


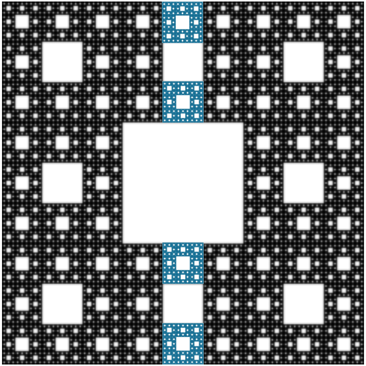
Let T_n be the vertical strip of width 3^{-n}.

Counterexample



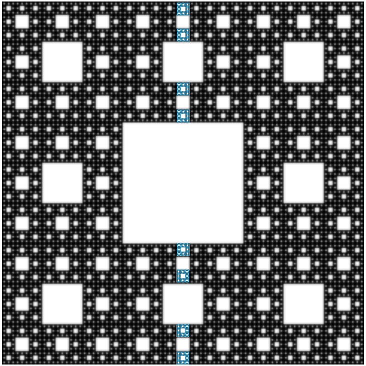
T_1

Counterexample



T_2

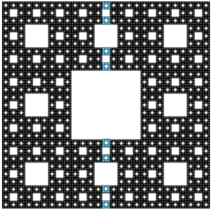
Counterexample



T_3

Counterexample

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq C\, r \Big(\fint_{B(x,r)} g^p d\mu \Big)^{1/p}$$



T_n

Define $f_n \in \mathrm{LIP}(S_3)$ such that $\fint_{S_3} |f_n - (f_n)_{S_3}| d\mu > C$ but

$$\int_{S_3} \mathrm{lip}(f_n) d\mu = 3^n \cdot \mu(T_n) = 3^n \cdot \frac{2^n}{8^n} \to 0 \ (n \to \infty)$$

Counterexample

- (S_3, d, μ) does not admit any p-PI

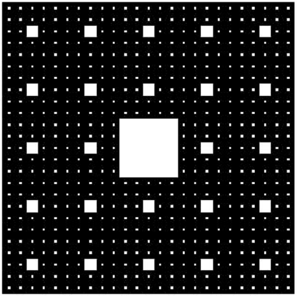
Bourdon-Pajot 02 Let (X, d, μ) be a bounded metric measure space with μ doubling and p-PI, and let $f : X \longrightarrow I$ be a surjective Lipschitz function from X onto an interval $I \subset \mathbb{R}$. Then, $\mathscr{L}^1_{|I} \ll f_{\#}\mu$. Here $f_{\#}\mu$ denotes the push-forward measure of μ under f.

Proof.

Let f be the projection on the horizontal axis. It can be checked that $f_{\#}\mu \perp \mathscr{L}^1$. □

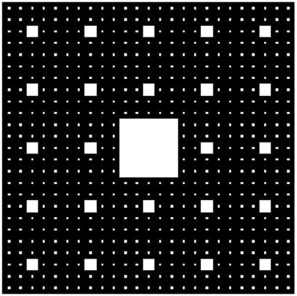
Question Higher dimensions?

Generalized Sierpinski carpets: $S_{\mathbf{a}}$



$$\mathbf{a} = (a_1^{-1}, a_2^{-1}, \ldots) \in \left\{ \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots \right\}^{\mathbb{N}}$$

Generalized Sierpinski carpets: $S_{\mathbf{a}}$



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For $\quad \mathbf{a} = \left(\dfrac{1}{a}, \dfrac{1}{a}, \dfrac{1}{a}, \dots\right)$ a odd,

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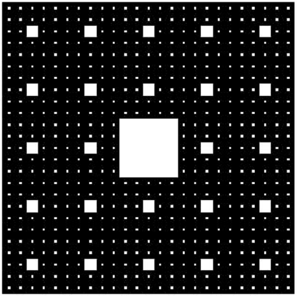

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- $S_{\mathbf{a}}$ does not admit any $p-$PI

Mackay, Tyson, Wildrick 13

- $(S_{\mathbf{a}}, d, \mu)$ supports a 1-PI if and only if $\mathbf{a} \in \ell^1$
- $(S_{\mathbf{a}}, d, \mu)$ supports a p-PI for some $p > 1$ if and only if $\mathbf{a} \in \ell^2$

Which is the role of the exponent p?

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \le Cr \Big(\fint_{B(x,\lambda r)} g^{p} d\mu \Big)^{1/p}$$

Hölder inequality: p-PI$\Longrightarrow$$q$-PI for $q \ge p$

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Federer-Fleming, Mazýa 60 Miranda 03

$(\mathbb{R}^n)$ $p = 1 \iff$ Isoperimetric inequality

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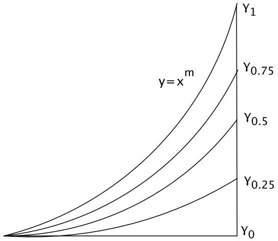

Example

$X := \{(x, y) \in \mathbb{R}^2 \, : \, x \ge 0, 0 \le y \le x^m\}$

$(X, |\cdot|, \mathscr{L}^2{}_{|X})$ X has $p-$PI $\iff$
$p > m + 1$

What happens when $p \to \infty$?

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What happens when $p \to \infty$?

$$\fint_{B(x,r)} |f - f_{B(x,r)}|\, d\mu \le Cr \Big(\fint_{B(x,\lambda r)} g^{p}\, d\mu \Big)^{1/p}$$

Hölder inequality: p-PI$\Longrightarrow$$q$-PI for $q \ge p$

Definition

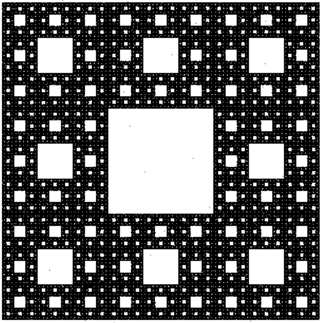
(X, d, μ) supports a weak ∞-Poincaré inequality if there exist constants $C > 0$ and $\lambda \ge 1$ such that for every function $f : X \to \mathbb{R}$ and every upper gradient g of f, the pair (f, g) satisfies

$$\fint_{B(x,r)} |f - f_{B(x,r)}|\, d\mu \le Cr \|g\|_{L^\infty(B(x,\lambda r))}$$

$\forall\, B(x, r) \subset X$.

$$\left.\begin{array}{r} X \text{ complete and } \infty\text{-PI} \\ \mu \text{ doubling} \end{array}\right\} \Longrightarrow X \text{ is quasiconvex}$$

$\boxed{\not\Leftarrow}$



Sierpiński carpet

- $d = d_{e|X} \quad \mu = \mathcal{H}^s, s = \frac{\log 8}{\log 3}$
- (X, d) is quasiconvex
- (X, d, μ) does not admit any p-PI, $1 \leq p \leq \infty$

And a differentiable structure?

Modulus of a family of curves

Definition

Let $\Gamma \subset \Upsilon = \{$non constant rectifiable curves of $X\}$ and $1 \leq p \leq \infty$. For $\Gamma \subset \Upsilon$, let $F(\Gamma)$ be the family of all Borel measurable functions $\rho : X \to [0, \infty]$ such that

$$\int_\gamma \rho \geq 1 \ \text{ for all } \ \gamma \in \Gamma.$$

$$\mathrm{Mod}_p(\Gamma) = \begin{cases} \inf_{\rho \in F(\Gamma)} \int_X \rho^p \, d\mu, & \text{if } p < \infty \\ \inf_{\rho \in F(\Gamma)} \|\rho\|_{L^\infty}, & \text{if } \boxed{p = \infty} \end{cases}$$

If some property holds for all curves $\gamma \in \Upsilon \backslash \Gamma$, where $\mathrm{Mod}_p \Gamma = 0$, then we say that the property holds for $p-$a.e. curve.

Remark

Mod_p is an outer measure

Lemma

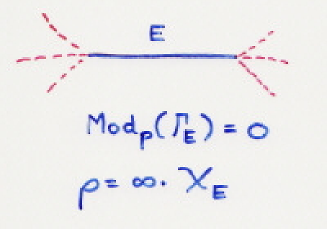
Let $\Gamma \subset \Upsilon$ and $1 \leq p \leq \infty$. The following conditions are equivalent:

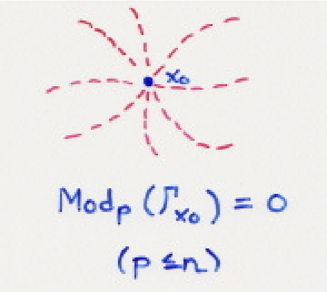
(a) $\operatorname{Mod}_p \Gamma = 0$.

(b) *There exists a Borel function $0 \leq \rho \in L^p(X)$ such that $\int_\gamma \rho = +\infty$, for each $\gamma \in \Gamma$ and $\|\rho\|_{L^\infty} = 0$.*

Examples

$\mathbb{R}^n, n \geq 2$





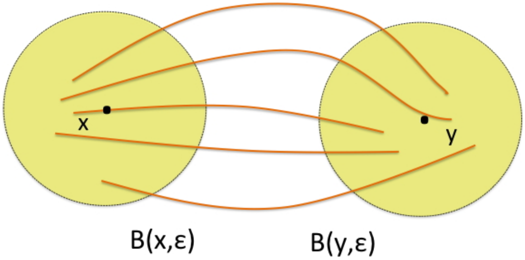
p-"thick" quasiconvexity

Definition

(X, d, μ) is a p-"thick" quasiconvex space if there exists $C \geq 1$ such that $\forall\, x, y \in X$, $0 < \varepsilon < \frac{1}{4}d(x,y)$,

$$\mathrm{Mod}_p(\Gamma(B(x,\varepsilon), B(y,\varepsilon), C)) > 0,$$

where $\Gamma(B(x,\varepsilon), B(y,\varepsilon), C)$ denotes the set of curves $\gamma_{p,q}$ connecting $p \in B(x,\varepsilon)$ and $q \in B(y,\varepsilon)$ with $\ell(\gamma_{p,q}) \leq Cd(p,q)$.



Geometric characterization: $p = \infty$

D-C, Jaramillo, Shanmugalingam 11

Let (X, d, μ) be a complete metric space with μ doubling. Then,

$$\boxed{X \text{ is } \infty\text{-“thick” quasi-convex} \iff X \text{ admits } \infty\text{-PI}}$$

Analytic characterization

D-C, Jaramillo, Shanmugalingam 11 Let (X, d, μ) be a connected complete metric space supporting a doubling Borel measure μ. Then

$$\boxed{\mathrm{LIP}^\infty(X) = N^{1,\infty}(X) \text{ with c.e.s.} \iff X \text{ admits } \infty\text{-PI}}$$

Geometric implications of p-PI

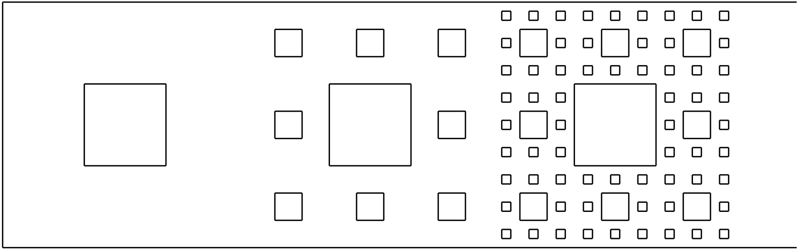
$$\left.\begin{array}{r} X \text{ complete and } p\text{-PI} \\ \mu \text{ doubling} \end{array}\right\} \Longrightarrow X \text{ is } p\text{-“thick” quasiconvex}$$

Remarks

- p-“thick” quasiconvex $\Longrightarrow$ quasiconvex
- The characterization is no longer true for $p < \infty$ $\boxed{\not\Leftarrow}$

A counterexample

$\mu = \sum_j \chi_{Q_j} \cdot \mu_j$ doubling measure



- X is p-thick quasi-convex $1 \leq p \leq \infty \implies \infty$-PI
- X admits an ∞-PI but does not admit any p-PI ($1 \leq p < \infty$)

$p-$Poincaré inequalities

D-C, Shanmugalingam 13+n, $n \in \mathbb{N}$ Let (X, d, μ) be a connected complete Ahlfors Q-regular space. Then the following conditions are equivalent:

(1) X supports a p-Poincaré inequality for some $p > Q$.

(2) There are constants $C > 0, \tau > 1$ such that every $u \in N^{1,p}(X)$ is $(1 - \frac{Q}{p})$-Hölder continuous.

(3) There is a constant $C > 0$ such that whenever $x_0, y_0 \in X$,

$$\mathrm{Mod}_p(\Gamma_{x_0,y_0}) \geq \frac{C}{d(x_0, y_0)^{p-Q}},$$

where Γ_{x_0,y_0} denotes the family of C-quasiconvex curves connecting x_0 to y_0.

Persistence of p-PI under GH-limits

Cheeger 99

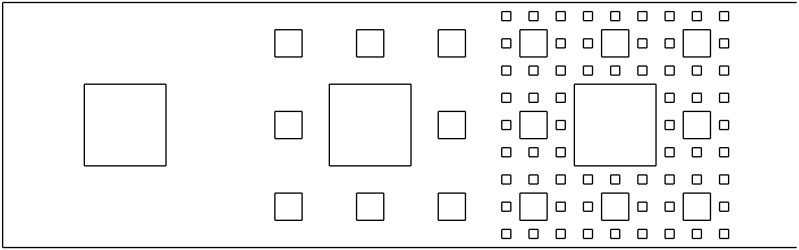
If $\{X_n, d_n, \mu_n\}_n$ with μ_n doubling measures supporting a p-PI $\boxed{p < \infty}$ (with constants uniformly bounded), and $\{X_n, d_n, \mu_n\}_n \xrightarrow{G-H} (X, d, \mu)$, then (X, d, μ) has μ doubling and supports a p-PI.

Corollary

The ∞-PI is non-stable under measured Gromov-Hausdorff limits.

Not Self-improvement of ∞-PI

Keith-Zhong 08 If X is a complete metric space equipped with a doubling measure satisfying a p-Poincaré inequality for some $1 < p < \infty$, then there exists $\varepsilon > 0$ such that X supports a q-Poincaré inequality for all $q > p - \varepsilon$.



Cheeger Analysis

Definition

A measurable differentiable structure on (X, d, μ) is a countable collection $\{(X_\alpha, \mathbf{x}_\alpha)\}_\alpha$ of measurable sets $X_\alpha \subset X$ and Lipschitz coordinates $\mathbf{x}_\alpha = (x_\alpha^1, \ldots, x_\alpha^{N(\alpha)}) : X \longrightarrow \mathbb{R}^{N(\alpha)}$ such that

(i) $\mu\left(X \setminus \bigcup_\alpha X_\alpha\right) = 0$;

(ii) $\exists N \geq 0$ (dimension) such that $N(\alpha) \leq N$ for each $(X_\alpha, \mathbf{x}_\alpha)$;

(iii) If $f : X \to \mathbb{R}$ is Lipschitz, then for each $(X_\alpha, \mathbf{x}_\alpha)$ there exists a unique (up to a set of zero measure) map $d^\alpha f \in L^\infty(X_\alpha; \mathbb{R}^{N(\alpha)})$ such that

$$\lim_{\substack{y \to x \\ y \neq x}} \frac{|f(y) - f(x) - d^\alpha f(x) \cdot (\mathbf{x}_\alpha(y) - \mathbf{x}_\alpha(x))|}{d(y, x)} = 0$$

for μ-a.e. $x \in X_\alpha$.

Cheeger Analysis

Cheeger differential of f (linear operator)

$$f \mapsto df := \sum_{\alpha=1}^{\infty} \chi_{X_\alpha} \cdot d^\alpha f \qquad |df(x)| = \operatorname{Lip} f(x)$$

- By Rademacher Euclidean spaces: single coordinate chart $(\mathbb{R}^n, \mathbf{x})$, with $\mathbf{x} = (x_1, \cdots, x_n)$ coordinate functions, $df = \nabla f$
- By Pansu 89: Carnot groups $\mathbb{R}^3 = \{(x, y, z)\}$ with Carnot-Carathèodory metric, $\mathbf{x} = (x_1, x_2)$, $df = \nabla_H f$.

Measured differentiable structures

X complete, μ doubling

Cheeger 99

$$\left.\begin{array}{r} X \text{ supports } p\text{-PI} \\ 1 \leq p < \infty \end{array}\right\} \Longrightarrow X \text{ admits a ``differentiable structure''}$$

Where is the p?

Keith 04

X satisfies Lip-lip $\Longrightarrow$ X admits a“differentiable structure”

Lip-lip condition

"The infinitesimal behaviour at a generic point is essentially independent of the scales used for the blow-up at that point"

Definition

X satisfies Lip-lip if $\exists\, C > 0$ such that $\forall f \in \mathrm{LIP}(X)$,

$$\mathrm{Lip} f(x) \leq C \,\mathrm{lip} f(x) \quad \mu\text{-}a.e.x$$

Here,

$$\mathrm{Lip} f(x) := \limsup_{r \to 0} \sup_{0<d(y,x)<r} \frac{|f(y) - f(x)|}{r},$$

and

$$\mathrm{lip} f(x) := \liminf_{r \to 0} \sup_{0<d(y,x)<r} \frac{|f(y) - f(x)|}{r}.$$

Remark

If $\mu \sim \lambda \implies (X, d, \mu)$ has Lip-lip iff (X, d, λ) has Lip-lip

Lip-lip condition

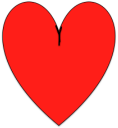
Keith 04

$$\left.\begin{array}{r} X \text{ complete and } p\text{-PI} \\ \mu \text{ doubling} \end{array}\right\} \Longrightarrow X \text{ has the Lip-lip condition}$$

Lip-lip condition

Keith 04

$$\left.\begin{array}{r} X \text{ complete and } p\text{-PI} \\ \mu \text{ doubling} \end{array}\right\} \Longrightarrow X \text{ has the Lip-lip condition}$$

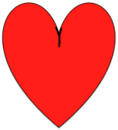


$\not\Longleftarrow$

Lip-lip condition

Keith 04

$$\left.\begin{array}{r} X \text{ complete and } p\text{-PI} \\ \mu \text{ doubling} \end{array}\right\} \Longrightarrow X \text{ has the Lip-lip condition}$$



$\nLeftarrow$

Proof.

For μ-a.e.x,

$$\frac{1}{C}\operatorname{Lip} f(x) \le \limsup_{r\to 0} \frac{1}{r} \fint_{B(x,r)} |f - f_{B(x,r)}|\, d\mu$$

$$\le L \limsup_{r\to 0} \left(\fint_{B(x,r)} \operatorname{lip} f(x)^p d\mu \right)^{\frac{1}{p}} = L \operatorname{lip} f(x)$$

$\square$

And $\infty-$PI?

$$\left.\begin{array}{r} X \text{ complete and } \infty\text{-PI} \\ \mu \text{ doubling} \end{array}\right\} \overset{NO\ a.s.}{\Longrightarrow} X \text{ has the Lip-lip condition}$$

Bate 12, Gong 12

$$\left.\begin{array}{r} X \text{ satisfies } \sigma-\text{Lip-lip} \\ \mu \text{ pointwise doubling} \end{array}\right\} \Longleftrightarrow X \text{ admits a Cheeger MDS}$$

Alberti representations

- $\Gamma(X)$ set of biLipschitz $\gamma : A \subset \mathbb{R} \to X$, A compact
- $\mathcal{P}$ probability measure on $\Gamma(X)$
- $\forall \gamma \in \Gamma(X)$, μ_γ Borel measure on X with $\mu_\gamma \ll \mathcal{H}^1_{|\gamma}$

Definition

$(\mathcal{P}, \{\mu_\gamma\}_\gamma)$ is an Alberti representation of μ if for each Borel set $B \subset X$

$$\mu(B) = \int_{\Gamma(X)} \mu_\gamma(B) d\mathcal{P}$$

Bate12 Cheeger MDS $\iff$ $\exists$ "collection" of AR

Alberti representations

- By Fubini's theorem there exists n independent Alberti representations of Lebesgue measure
- For $f : \mathbb{R}^n \to \mathbb{R}$ Lipschitz and γ curve (from AR), $f \circ \gamma$ is Lipschitz therefore differentiable a.e. (Lebesgue)
- By combining such derivatives from each AR one prove that partial derivatives exists a.e.
- Partial derivatives form a derivative a.e.

Bate 12 Let μ be a Radon measure on $(\mathbb{R}^n, |\cdot|)$, $n = 1, 2$. Every Lipschitz function is differentiable μ−a.e. $\iff \mu \ll \mathscr{L}^n$.

Derivations

Definition

Weaver 99 A bounded linear operator

$$\delta : \mathrm{Lip}_b(X) \to L^\infty(X, \mu)$$

is called a (metric) derivation if it satisfies

- *product rule*: $\delta(fg) = f\,\delta g + g\delta f \quad \forall f, g \in \mathrm{Lip}^\infty(X)$;
- *weak-star continuity*: if $f_i \overset{*}{\rightharpoonup} f$ in $\mathrm{Lip}^\infty(X) \Rightarrow \delta f_i \overset{*}{\rightharpoonup} \delta f$ in L^∞_μ

$\boxed{\Upsilon(X, \mu)}$ space of derivations with respect to μ on X.

We call a set $\{\delta_i\}_{i=1}^k$ linearly dependent in $\Upsilon(X, \mu)$ if there exist $\{\lambda_i\}_{i=1}^k$ in $L^\infty(X, \mu)$, not all zero, so that

$$\lambda_1\delta_1 + \cdots + \lambda_k\delta_k \;=\; 0 \quad \text{rank}\, k.$$

Gong 12 (X, d, μ) admits a non-trivial basis of derivations $\iff$ Cheeger MDS

Gong 13 Let μ be a Radon measure on $(\mathbb{R}^n, |\cdot|)$. Then $\Upsilon(X,\mu)$ has rank-n $\iff$ $\mu \ll \mathscr{L}^n$. Moreover, derivations with respect to μ are linear combinations of the differential operators $\{\partial/\partial x_i\}_{i=1}^n$ with scalars in $L^\infty(\mathbb{R}^n, \mu)$.

D-C, Gong, Jaramillo 13 a.s Sierpiński fractals are differentiably trivial with respect to the Euclidean metric.

So...

$$\boxed{\mu \text{ doubling}}$$

$$\begin{tikzcd}
\begin{array}{c} p\text{-PI} \\ 1 \le p < \infty \end{array} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathrm{Lip}\text{-}\mathrm{lip} \arrow[d, Leftrightarrow] \\
\infty\text{-PI} \arrow[r, phantom, "\not\Longrightarrow"] & \text{Cheeger MDS} \arrow[d, Leftrightarrow] \\
& \text{AR measure} \arrow[d, Leftrightarrow] \\
& \text{Basis of derivations}
\end{tikzcd}$$

Merci pour votre attention!

Isoperimetric inequality

The isoperimetric problem is to determine a plane figure of the largest possible area whose boundary has a specified length.

Let us consider the inequality:

$$\left(\int_\Omega |u|^q d\mu\right)^{1/q} \leq C \int_\Omega |\nabla u| d\mu,$$

where $q \geq 1$, $\Omega \subset \mathbb{R}^n$ open, μ is a measure and $u \in C_0^\infty(\Omega)$.

Federer-Fleming, Mazýa 1960

The inequality is satisfied with $q \geq 1$ if and only if

$$(\mu(\Omega))^{\frac{1}{q}} \leq C\mathscr{H}^{n-1}(\partial\Omega),$$

for each open and bounded domain Ω with smooth boundary.

Volver