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Convex trace functions with applications in Q physics.

results from April 2013.

f convex onto $\rightsquigarrow$ perspective f^n $g(u,t) = f(t^{-1}u)t$ on L is also a convex f^n of 2 variables.

$\sigma_f(A,B) = B^{1/2} f(B^{-1/2} A B^{-1/2}) B^{1/2}$ = nc perspective f^n

where A, B are p.d. studied with Löwner rep th by Ando.

Another proof by Effros: If f is op. convex, σ_f is op. convex

t log t is op cvx $g(t,s) = t\log t - t\log s$ is op. convex. Here this amounts to a commutative result

In fact $(A,B) \mapsto \mathrm{tr}\, K^* g(L_A, R_B)(K)$ convex for all K.

3 applications: ① residual entropy:

$\varphi(A_1, \ldots, A_k) = -\mathrm{tr}\, A\log A + \sum_i \mathrm{tr}\, A_i \log A_i$ is a convex f^n.

entropy of the total system

entropy of the subsystems

$\downarrow$

in fact, can be written as a sum of partial entropies, so this is not a deep thm.

② Nonhappy gain over q channels = c.p. trace preserving map ϕ

$= A \mapsto \sum_i a_i^* A a_i$

$a_1 a_1^* + \ldots + a_k a_k^* = 1$.

$S(\phi(A)) - S(A)$ is a convex f^n!

$\phi(A_1, \ldots, A_k) = S(\phi_1(A_1) + \ldots + \phi_k(A_k)) - \sum_i S(A_i)$ for k q channels.

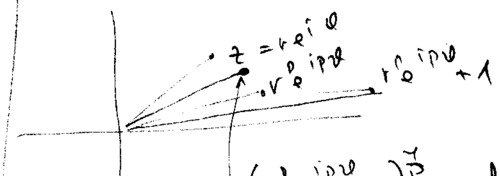
$\longrightarrow$ use block matrices $\begin{pmatrix} A & \\ & A \end{pmatrix}$, $K = \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix}$

Carlen-Lieb's trace f^n: $(A,B) \mapsto \mathrm{tr}\left[(A^p+B^p)^{1/r}\right]$, $0<p\le r\le 1$.
is concave in p.d. matrices.

- simple case: $r = p$:
- What about $p \in [1,2]$: also, but proofs are difficult.

$t \mapsto (t^p+1)^{1/p}$ is op. monotone.

→ cf. Löwner: must allow an analytic continuation into $\mathbb{C}^+$



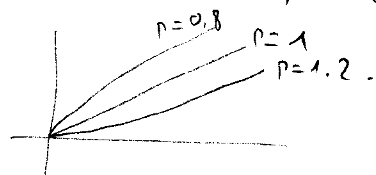
$(r^p e^{ip\vartheta}+1)^{1/p}$, where $\underset{>0}{\arg} < \vartheta < \pi$. | this means f is operator monotone concave.

We can prove CL by perspective f^n: $(t^p+s^p)^{1/p}$: it is op concave,
so is also $(t,s) \mapsto (t^p+s^p)^{1/r}$ by composing with $t \mapsto t^{p/r}$.
Then $(A,B) \mapsto K^*\left((L_A^p + R_B^p)^{1/r}\right)(K)$ is concave $\frac{p}{r} \le 1$.
Thus ($K=1$), $\mathrm{tr}\,(A^p+B^p)^{1/r}$ is.

→ Thus we have 2 slides proof.

→ $p \in [1,2]$: the same construction works. We need to prove that $f(t) = (t^p+1)^{1/p}$ is op. monotone. There is a problem. We do not have a simple χ^{tion} of op. monotone f^n... You lose track.

Another idea.
$f(t) = (t^p+1)^{1/p}$:



Rep. measure for f: $0<p\le 1$
$f(re^{i\vartheta}) = (r^p e^{ip\vartheta}+1)^{1/p}$. Let arg be determined in $[0,2\pi[$
$A_p(r,\vartheta)$ angle between $\mathbb{R}^+$ and $(r^p e^{ip\vartheta}+1)^{1/p}$ is $A_p(r,\vartheta) = \frac{1}{p}\arg(r^p\cos p\vartheta + 1 + i r^p \sin p\vartheta)$
$0 < A_p(r,\vartheta) < \vartheta < \pi$ for $r>0$, $0<\vartheta<\pi$..

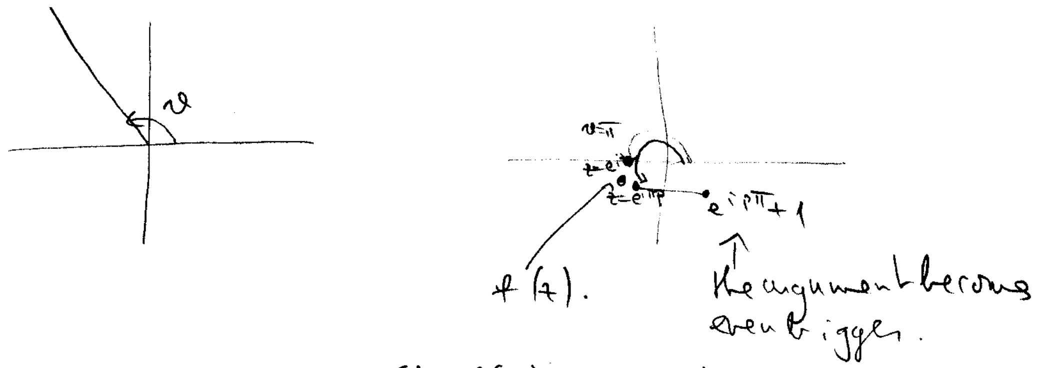
$\mathrm{Im} f(re^{i\vartheta}) = \dots$ rep measure $= \frac{1}{\pi} \lim_{\vartheta \to \pi} \mathrm{Im} f(re^{i\vartheta})$

$(t^p+1)^{1/p} = \alpha + \beta t + \int_0^\infty \left(\frac{\lambda}{1+\lambda^2} - \frac{1}{t+\lambda}\right) h_p(\lambda)\,d\lambda .$

$\quad \frac{1}{\pi}(1+\lambda^{2p}+2\lambda^p\cos p\pi)^{\frac{1}{2p}} \sin A_p(\lambda,\pi).$
weight f^n.

Then we have $\pi < A_p(\lambda,\pi) < 2\pi$

Picture for $p>1$:



$\pi < \mathrm{Arg} f(t) < 2\pi$. !

The really fal formula: the rep. measure is an analytic fⁿ in p between 0 and 2.
But $(t^p+1)^{\frac{1}{p}}$ also is. And they coincide between 0 and 1.
Therefore everywhere! But we have sin( ) < 0. And it
multiples an op. concave fn
Thus it is op. convex!

But what between 2 and 3. There it does not work anymore, even if the formula still makes sense, but RHS is no longer analytic. (Arg crosses $\mathbb{R}^+$)
And the theorem is also false...

CL drew on the heat eqⁿ.

Variational ineq. $(x^p+y^p)^{\frac{1}{p}} \le \lambda^{\frac{p-1}{p}} x + (1-\lambda)^{\frac{p-1}{p}} y$.

eq for $\lambda = \frac{x^p}{x^p+y^p}$.

The same holds for p.d. matrices... X between 0 and 1.

Proof. Fréchet differential of $\varphi(X,Y) = \mathrm{Tr}(X^p+Y^p)^{\frac{1}{p}}$.
$d\varphi(X,Y)(A,B) \geq \varphi(A,B)$ for p.d. X, Y, A, B, because φ is concave and pos. homogeneous.
... chain rule... Under the trace, you can consider $\mathrm{Tr}\, df = \mathrm{Tr} f'$!!!