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Estimate of Szlenk index

<u>Motivation</u>: dentability / fragmentability: Let B be a bounded subset of a B-space X. Let us compare weak and norm topology. Does there exist $x \in B$ that admits $U \ni x$ weakly open with arbitrarily little diameter?

<u>Def</u>: $B \subset X$ is fragmentable if $\forall \varepsilon > 0\ \exists x \in B\ \exists U \ni x$ w-open, $\operatorname{diam} U < \varepsilon$.
— dentable — slice —

<u>Def</u>: X has RNP if every $B \subset X$ bounded is dentable.

For a dual space, consider the same notions using w^*-topology.

<u>Theorem</u>: (Namioka-Phelps-Stegall) X^* is RNP iff X^* is w^*-fragmentable iff X^* is w^*-dentable iff X is Asplund [if X is sep, this is equivalent to X^* sep.]

Def: For $B \subset X^*$, let $S_\varepsilon(B) = \{x \in B : \forall U \ni x\ w^*\text{-open}, \operatorname{diam} U \geq \varepsilon\}$,

$$S_\varepsilon^2(B) = S_\varepsilon^1 \circ S_\varepsilon^1(B), \text{ etc.}, \quad S_\varepsilon^\omega = \bigcap_n S_\varepsilon^n.$$

Then either $S_\varepsilon^\alpha(B) = \emptyset$ or $S_\varepsilon^{\alpha+1}(B) = S_\varepsilon^\alpha(B)$.

If such an α exists, let the infimum of such be the <u>ε-Szlenk index</u>.

The <u>Szlenk index</u> of X is the least upper bound of the $S_\varepsilon^\alpha(B)$ for $\varepsilon > 0$.

Then there is also $Dz(X)$: the dentability index. What is the relationship between α and β

<u>Theorem</u>: (Lancien, Bossard) There is $\varphi : \omega_1 \to \omega_1$ such that for all sep. X,
$Dz(X) \leq \varphi(Sz(X))$.

Tool: Descriptive set theory, no explicit bounds!

Theorem: (Raja) $\forall X$ $Dz(X) \leq \omega^{Sz(X)}$: geometric tools.

N.B.: The Szlenk index is an isomorphic invariant!

Purpose: there is no sep. refl. space containing all other such spaces.
— Because there is a scale of refl. sep. spaces with increasing Sz index.

- $Sz(C(K))$ has been computed for all metrisable compact spaces.

  $C[0,\omega^{\omega^\alpha}]$ for $\alpha<\omega_1$ is a complete scale! (up to isomorphism)

  $Sz(C[0,\omega^{\omega^\alpha}])=\omega^{\alpha+1}$ (Samuel).

Theorem: Dz is also an invariant and $Dz(C[0,\omega^{\omega^\alpha}])=\begin{cases}\omega^{\alpha+2} & \text{if } \alpha \text{ finite}\\ \omega^{\alpha+1} & \text{if } \alpha \text{ infinite}\end{cases}$

It is also known that $Sz(X)=\omega^\alpha$ for some α.

Theorem: $Sz(L_2(X))\geq Dz(X)$.

New theorem: Let X be sep: $Sz(X)=\omega^\alpha$ for some α. Then $Sz(L_2(X))\leq\begin{cases}\omega^{\alpha+1} & \text{finite}\\ \omega^{\alpha} & \text{infinite}\end{cases}$

Th [Zippin] Let X be separable and let $Sz(X,\frac{\varepsilon}{8})\leq\beta$, $\varepsilon>0$ given.

Then there are Y,Z s.t. $X,Y\hookrightarrow Z$ and $Y\cong C[0,\omega^{\beta+1}]$ and s.t. $d(x,Y)\leq\varepsilon\|x\|$ for every $x\in X$.

Alternative description of $Sz(X)$ using trees in X:

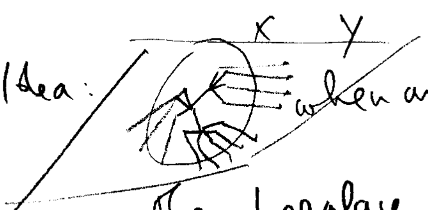
$(T,\leq)=\{(x_1,\dots,x_n): x_i\in X, n\in\mathbb{N}\}$ and $(x_1,\dots,x_n)\leq(x_1,\dots,x_n,x_{n+1},\dots)$

Derivation for trees: $S(T)=\{(x_1,\dots,x_n)\in T:\exists\,(x_1,\dots,x_n,x_{n+1})\in T\}$.

(Alspach, Judd, Odell): there is a notion of K-ℓ_1^+-weakly null trees in X.

Important properties:
- $(x_1),(x_1,x_2),\dots,$ form a K-ℓ_1^+ average:
- $\|\sum_i\alpha_i x_i\|\geq\frac{1}{K}\sum_i\alpha_i$ for positive α_i
- each nonterminal node has ∞-ly many followers; and they form a w-null sequence of followers
- $\frac{1}{K}$ and ε are connected.

Theorem. $Sz(X)=\sup_{K>0}o(T)$ ← height of the tree T index.

Idea: 

when are there large trees with many branches.

Then I replace nodes with vectors: this gives me a chance to transfer a tree into sthg. living in Y: $(x_1,\dots,x_n)$ becomes $(y_1,\dots,y_n)$

As the ℓ^1 is the strongest norm, perturbing an ℓ^1-basis still $\|x_i-y_i\|\leq\varepsilon$ gives an ℓ^1-basis!

Summary: there is a way to reduce to a $C(K)$ space.

To finish the argument: $Sz(X) \dots Dz(X) \dots Sz(L^2(X))$ is used to estimate $Dz(X)$. Then compare $\underset{L_2(X)}{\text{with}}$ $L_2(\underbrace{C([0,\omega^\beta])}_{Y})$.

"If X metrically close to Y, then $L^2(X)$ metrically close to $L^2(Y)$".

then generate large trees in $L_2(C[0,\omega^\beta])$.
In the end use Lancien-Prochazka.

Remark: The argument uses $Sz(X,\varepsilon)$ ... we get the final estimate for all sep. X even if $Sz(X) \neq Sz(C[0,\omega^{\omega^\alpha}])$ for all α.

→ here the argument works even for spaces with Szlenk index that are not of the kind $Sz(C[0,\omega^{\omega^\alpha}])$. This is because we work on $Sz(X,\varepsilon)$.

there was another argument - brute force proof.