<u>q-Deformation of free independence.</u>

q comes from <u>q-Fock space</u> $\mathcal{F}_q(H)$, $-1 \le q \le 1$, introduced by Bożejko and Speicher:

$$a_f^- a_g^+ - q\, a_g^+ a_f^- = \langle f, g \rangle I \qquad (\text{q-CCR relations})$$

and <u>q-Brownian motion</u> $\{Q_t\}_{t \ge 0}$, $Q_t = a_f^+ + a_f^-$, $f \in H = L^2(\mathbb{R}^+)$, $f = \chi_{[0,t[}$

For $q = 1$, this gives the symmetric Boson space, classical probability – "tensor" independence

$q = 0$, free probability theory – freeness, "free independence".

Q: interpolation between these two cases: "q-independence"

<u>What is "independence"?</u> Let (A, φ) be a n.c. prob. sp. (algebra, linear functional)

Commutative counterpart: $(\Omega, \mathcal{F}, P) \rightsquigarrow L^\infty(\Omega)$, $\varphi(\cdot) = \int_\Omega \cdot \, dP$.

Let $x, x', x'' \ldots \in A_1$, $y, y', y'' \ldots \in A_2 \subseteq (A, \varphi)$: <u>tensor independence</u> is, for A_1, A_2, $\varphi(x y x' y') = \varphi(x x')\,\varphi(y y')$

<u>free</u> independence is $\langle x y x' y' \rangle = \langle x x' \rangle \langle y \rangle \langle y' \rangle + \langle x \rangle \langle x' \rangle \langle y y' \rangle - \langle x \rangle \langle x' \rangle \langle y \rangle \langle y' \rangle$

where $\langle \; \rangle = \varphi(\cdot)$.

Example: $x y x' y'$ gives rise to a partition of the n-point set, $n = 4$.

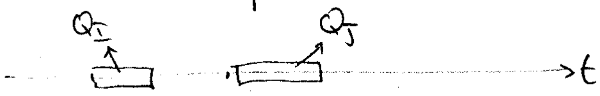
Thus, the last formula is $\langle \pi \rangle = 1 \cdot \langle \sigma_1 \rangle + 1 \cdot \langle \sigma_2 \rangle + (-1) \langle \sigma_3 \rangle$, where $1 = t(\pi;\sigma_1)$, $1 = t(\pi;\sigma_2)$, $-1 = t(\pi;\sigma_3)$

<u>Independence</u> is therefore to give $\mathcal{Q} = \{ t(\pi : \sigma) \}$, a universal calculation rule for mixed moments

$$\varphi[\underset{A_{l_1}}{x_1} \cdots \underset{A_{l_n}}{x_n}] = \sum_{\sigma \le \pi} t(\pi : \sigma) \prod_{V \in \sigma} \varphi_V(x_1, \ldots, x_n)$$

$(l_1 \ldots l_n) = \pi$ $\qquad$ σ is a refinement of π.

Impossibility of q-convolutions and therefore of q-independence: van Leeuwen–Maassen.



: there are functions $f(x)$, $g(x)$ s.t.

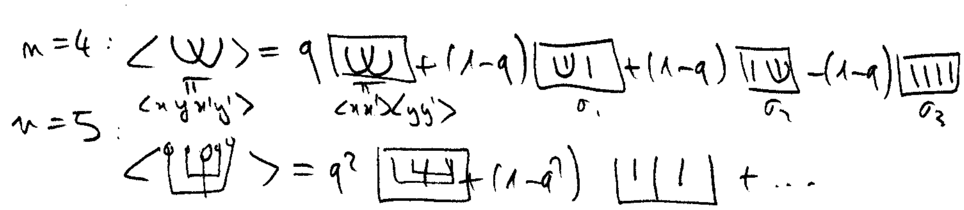
(here, if $I =]s,t]$), $Q_I = Q_t - Q_s$.

$Q_I \overset{\text{dist}}{=} f(Q_I)$, $Q_J \overset{\text{dist}}{=} g(Q_J)$ but $Q_I + Q_J \overset{\text{dist}}{\ne} f(Q_I) + g(Q_J)$

Nevertheless, we have: <u>Theorem</u>: There is a universal calculation rule $\mathcal{Q} = \{A(\pi;\sigma)\}$ ($\langle X_i\rangle = 0$, $\langle X_i^2\rangle = 1$)
such that $\left\{ Y_t^{(n)} = \frac{X_1+\dots+X_{[nt]}}{\sqrt{n}} \right\}_{t\geq 0} \xrightarrow{\text{in joint moments}} \{Q_t\}_{t\geq 0}$ satisfy certain relations wrt $\mathcal{Q}$

Remark: the increments Q_I, Q_J are not "q-independent" if $I \cap J = \emptyset$!

Example of $\{t(\pi;\sigma)\}$:



Idea of proof: We are looking for $\mathcal{Q} = \overset{q\text{-independence}}{\underset{\mathbb{C}\xi_0 + H_0^0}{\{t(\pi;\sigma)\}}} \longleftrightarrow$ q-products: $\underset{\ell\in L}{\overset{(q)}{\star}} (A_\ell, \varphi_\ell)$

(A_ℓ, φ_ℓ): $\xi_\ell : A_\ell \to B(\overset{\circ}{H_\ell})$: then $\underset{\ell\in L}{\overset{(q)}{\star}} B(\mathbb{C}\xi_\ell \oplus \overset{\circ}{H_\ell})$ is the algebra generated by the images $\pi_\ell(B(H_\ell))$

then $\pi_\ell : \underset{\text{marginal space}}{B(H_\ell)} \longrightarrow \underset{\text{big space}}{B(K)}$

$q=0$: $K = \mathbb{C}\xi \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{\ell_1 \neq \dots \neq \ell_n} H^0_{\ell_1} \otimes \dots \otimes H^0_{\ell_n}$

For general q, we are looking for the q-~~free~~ product
This means that we look for a q-analogue of the condition $\ell_1 \neq \dots \neq \ell_n$.
But this condition corresponds to an orthogonal projection in $\mathcal{F}_q(H^0)$.
We are looking for some nice projections.

N.B.: this product is not associative
But we get an associative convolution law for prob. measures.
→ through cumulants.

N.B.: This construction is more general than Fock space