$\boxed{\text{P.R.I.}}$

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X Banach space
X^* dual

| How to construct a projection $P: X^* \to X^*$?

Consider (V, Y), where $V \subset\subset X$ and $Y \subset X^*$ and $\begin{array}{rcl} Y & \longrightarrow & V^* \\ x^* & \longmapsto & x^*|_V \end{array}$

Suppose 1°) R is surjective.
2°) R is injective. Consider $\begin{array}{rcl} P: X^* & \longrightarrow & X^* \\ x^* & \longmapsto & R^{-1}(x^*|_V) = Px^* \end{array}$; then

P is a projection $X^* \to X^*$ with $PX^* = Y$.

If also 3°) R is an isometry, then $\|P\| = 1$.

The Godefroy-Fabian theorem is based on this observation.

Let X be with $\|\ \|$ F smooth. $\begin{array}{rcl} \varphi: X & \longrightarrow & S_{X^*} \\ x & \longmapsto & \|\ \|'(x) \end{array}$ is continuous

By Bishop-Phelps: $\forall V \subset X$, let $Y = \overline{\operatorname{span}\, \varphi(V)}$, ... 1°) 2°) 3°) ...

will be verified considering $V_0 \subset V_1 \subset \dots$ separable, $V = \overline{\bigcup V_n}$.

<u>General case</u>: consider $\begin{array}{rcl} \partial: X & \longrightarrow & 2^{X^*} \\ x & \longmapsto & \{x^* \in S_{X^*} : \langle x^*, x\rangle = \|x\|\} \end{array}$

∂ is $\|\cdot\|-w^*$ u.s.c. (and not lsc, so that we cannot apply Michael's thm)

Then use ~~Lange~~ Jayne Rogers selection theorem [of Baire 1-class] and Simons' lemma (a pearl of functional analysis). Separable reduction: (before we could not go beyond $\aleph_1$)

Now there is a proof by Stegall without Simons's lemma!

$$\begin{array}{ccc} C(B_{X^*}) & \xrightarrow{\ \partial^{\text{red}}\ } & 2^{B_{X^*}} \\ \uparrow & & \| \\ X & \xrightarrow{\ \partial\ } & 2^{B_{X^*}} \end{array}$$

$\begin{array}{rcl} C(B_{X^*}) & \xrightarrow{\ \partial^{\text{red}}\ } & B_{X^*} \\ f & \longmapsto & \{x^* : f(x^*) = \max_{B_{X^*}} f\} \end{array}$

Problem: • $x^* \in X^*$ may not attain its norm
• $\|x\| = \langle x^*, x\rangle$

If X is separable, $\forall x^* \in B_{X^*}$ $\exists f \in C(B_{X^*})$ $\partial(f) = \{x^*\}$ = a singleton
$= \langle y^*, x\rangle$ with $\|y^*\| = \|x^*\|$.

$\to$ Lindelöf's property

Fact: If X is separable, $\forall x^* \in B_{X^*}\ \exists f \in C(B_{X^*})\ \partial(f) = \{x^*\}$.

$\rightarrow$ Proof: $\{x_1, x_2, \dots\} \subset B_X$; let $f(y^*) = 1 - \sum_{n=1}^{\infty} |\langle y^*, x_n\rangle - \langle x^*, x_n\rangle|$

then $1 = f(x^*) > f(y^*) \geq -1$ whenever $y^* \in B_{X^*} \setminus \{x^*\}$!

Now let $\mathcal{L}(S) = \{1 - \sum 2^{-n} |\langle \cdot, x_n\rangle - a_n| : x_1, \dots, x_k \in S,\ a_1, \dots, a_k \in [-1,1]\}$

Clearly $f \in \overline{\mathcal{L}(B_X)}$ (let $a_n = \langle x^*, x_n\rangle$)

Jayne-Rogers $\exists \lambda_j : C(B_{X^*}) \to X^*$ continuous $\lambda_j(f) \to \lambda_0(f)$

and $\lambda_0(f) \in \partial(f)$: $f(\lambda_0(f)) = \max f(B_{X^*})$ for all f.

<u>Prop. 2</u>: Let $(X, \|\cdot\|)$ Asplund. Then $B_{X^*} \subset \overline{\Lambda(\mathcal{L}(B_X))}$.

Then $B_{X^*} \subset \overline{\Lambda(\mathcal{L}(B_X))}$. More generally $\forall V \subset X$ $B_{V^*} \subset \overline{\Lambda(\mathcal{L}(B_V))}|_V$

$V := \overline{\operatorname{span} \Lambda(\mathcal{L}(B_V))} \ni x^* \mapsto x^*|_V =: R(x^*) \in V^*$ is <u>onto</u>.

Kubiś: projectional skeleton!

Th 4: Let $(X, \|\ \|)$ nonsep Asplund space. Then $\exists\ \mathcal{V}$ of subspaces of X and a family $\{Y_V : V \in \mathcal{V}\}$ of ss of X^* st.

(i) $\bigcup\{V : V \in \mathcal{V}$ and dens $V = \aleph\} = X$ for every $\aleph <$ dens X

$\bigcup\{Y_V :$ ———————— $\} = X^*$

... "projectional skeleton".

X^* sep ... space metrisable ... Urysohn's lemma.

Is there a space where the fact above does not hold?

N.B.: difference between skeleton and PRI: not a linearly ordered set anymore, but a σ-complete partially ordered one.