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Sobolev and Besov spaces on quantum tori

Def: A n.c. torus $\mathbb{T}_\theta^d$ is given by a skew-symmetric real matrix $\theta = (\theta_{kj})$: one defines the ~~this the~~ C^*-algebra generated by d unitaries $U_1, \dots, U_d$ s.t. $U_k U_j = e^{2\pi i \theta_{kj}} U_j U_k$;

a trace is defined for $x = \sum'_{m \in \mathbb{Z}^d} \alpha_m U^m$ by $\tau(x) = \alpha_0$: τ is positive, faithful, tracial.

Then $\mathbb{T}_\theta^d$ is the w^*-closure of the GNS construction of (A_θ, τ)

There is a differential structure on $\mathbb{T}_\theta^d$: $A_\theta^\infty = \{ \sum_m \alpha_m U^m : \alpha_m \text{ decreases rapidly} \}$ is viewed as $C^\infty(\mathbb{T}_\theta^d)$.

N.B.: The commutative torus corresponds to $\theta = 0$

Let ∂_j: $\partial_j(U_j) = 2\pi i\, U_j$, $\partial_j(U_k) = 0$ if $j \neq k$. We have $\tau \circ \partial_j = 0$, $\tau(\partial_j a\, b) = -\tau(a\, \partial_j b)$

If $x \in L_p(\mathbb{T}_\theta^d)$, we have $(\partial^\alpha x, u) = (-1)^{|\alpha|}(x, \partial^\alpha u)$ for $u \in A_\theta^\infty$, $\alpha \in \mathbb{N}^d$

Let $\Delta = \sum_{j=1}^d \partial_j^* \partial_j = -\sum_{j=1}^d \partial_j^2$: its spectrum ~~is~~ $\{4\pi |m|_2^2\}$, where $|m|^2 = (\sum m_i^2)^{\frac12}$.

the Sobolev space $W_p^k(\mathbb{T}_\theta^d) = \{ x \in L_p(\mathbb{T}_\theta^d) : \partial^\alpha x \in L_p(\mathbb{T}_\theta^d) \text{ for } |\alpha| \le k \}$.

$\|u\| = \sum \|\partial^\alpha x\|_p$

$$H_p^s(\mathbb{T}_\theta^d) = \{ x : (1+\Delta)^{\frac{s}{2}} x \in L_p \}.$$

To define the Besov space, let $\varphi \in S(\mathbb{R}^d)$, $\operatorname{supp} \varphi = \{ \xi : \frac12 \le |\xi| \le 2 \}$ with $\sum_{k \in \mathbb{Z}} \varphi(2^{-k}\xi) = 1$ for $\xi \neq 0$.

then ~~let~~ $\widehat{\varphi_k}(\xi) = \varphi(2^{-k}\xi)$ and $\hat{\varphi}(\xi) = 1 - \sum_{k=1}^\infty \varphi(2^{-k}\xi)$. then

$B_{p,q}^s(\mathbb{R}^d)$ is defined by $\|f\| = \|\varphi * f\|_p + \left(\sum'_k (2^{sk} \|\varphi_k * f\|_p)^q \right)^{\frac1q}$

Then $\|f\|_{B^s_{p,q}} = \|\tilde{\varphi} * f\|_p + (\dots \|\tilde{\varphi}_k * f\|_p)$ with $\tilde{\varphi} = \sum' \hat{\varphi}(k) e^{2\pi i k x}$ ~~$\tilde{\varphi}_k = \sum \hat{\varphi}_k(k) e^{2\pi i x}$~~

defines $B_{p,q}^s(\mathbb{T}^d)$

Similar definitions hold on $\mathbb{T}_\theta^d$: $\|x\| = \|\tilde{\varphi} * x\|_p + \left(\sum'_{k \in \mathbb{Z}} (\|\tilde{\varphi}_k * x\|_p 2^{ks})^q \right)^{\frac1q}$

et la convolution est définie par multiplication des coefficients de Fourier.

Théorème: ① $1 \le p, q < \infty$: $B_{p,q}^s(\mathbb{T}_\theta^d) \hookrightarrow B_{p_1,q}^{s_1}(\mathbb{T}_\theta^d)$ si $1 \le p < p_1 < \infty$ et $s - \frac{d}{p} = s_1 - \frac{d}{p_1}$

② si $1 < p < \infty$: $H_p^s(\mathbb{T}_\theta^d) \hookrightarrow H_{p_1}^{s_1}(\mathbb{T}_\theta^d)$ si ——"——

③ si $1 \le p < \infty$: $W_p^k(\mathbb{T}_\theta^d) \hookrightarrow W_{p_1}^{k_1}(\mathbb{T}_\theta^d)$ si $k - \frac{d}{p} = k_1 - \frac{d}{p_1}$.

C'est une application du principe de transfert : soit $N_\vartheta = L_\infty(\mathbb{T}^d) \bar{\otimes} \mathbb{T}_\vartheta^d$
$\nu = \int d\mu \otimes \tau$

de sorte que $L_p(N_\vartheta) \simeq L_p(\mathbb{T}^d ; L_p(\mathbb{T}_\vartheta^d))$

Si $x \in L_p(\mathbb{T}_\vartheta^d)$, on considère $\tilde{x} : \mathbb{T}^d \longrightarrow L_p(\mathbb{T}_\vartheta^d)$
$x = \sum \alpha_m U^m$
$z \longmapsto \pi_z(x) = \sum \alpha_m z^m U^m$.

On a $\|\tilde{x}\|_p = \|x\|_p$ et $H_p^s(\mathbb{T}_\vartheta^d) \subset H_p^s(\mathbb{T}^d, L_p(\mathbb{T}_\vartheta^d))$
et $W_p^k(\mathbb{T}_\vartheta^d) \subset W_p^k(\quad)$
et $B_{p,q}^s(\mathbb{T}_\vartheta^d) \subset B_{p,q}^s(\quad)$

N.B. : on n'a pas en général $B_{p,q}^s(\quad , L_p \quad) \hookrightarrow B_{p_1,q}(\quad , L_{p_1} \quad)$

Mais on a $W_p^k(\mathbb{T}_\vartheta^d) \simeq H_p^k(\mathbb{T}_\vartheta^d)$ si $p>1$ et
$W_p^k(\mathbb{T}^d, L_p(\mathbb{T}_\vartheta^d)) \simeq H_p^k(\mathbb{T}^d, L_p(\mathbb{T}_\vartheta^d))$ $\to$ espace UMD !

Idée de la preuve : noter que $\|x\|_{B_{p,q}^s} \approx \|\hat{x}(0)\| + \left(\int_0^1 (1-r)^{(k-s)q} \left\| \frac{\partial^k P_r[x]}{\partial r^k} \right\|_p^q \frac{dr}{1-r}\right)^{\frac{1}{q}}$
si $k > \max(s,0)$ et $P_r[x] = \sum_m \hat{x}(m) r^{|m|_2} U^m$
est le noyau de Poisson.

Ds la preuve classique, pour $B_{p,q}^s(\mathbb{R}^d, \nu)$, ~~###~~
$\|f\|_{B_{p,q}^s(\mathbb{R}^d)} \approx \|\varphi_0 * f\|_p + \left(\int_0^1 \varepsilon^{(k-s)q} \left\| \frac{\partial^k \varphi_\varepsilon * f}{\partial \varepsilon^k} \right\|_p^q \frac{d\varepsilon}{\varepsilon}\right)^{\frac{1}{q}}$ avec $\varphi_\varepsilon = C_d \frac{\varepsilon}{(\varepsilon^2 + |s|^2)^{d/2}}$

Pour démontrer (2), rappelons que $B_{p,r}^{s^*} = (B_{p,q_0}^{s_0}, B_{p,q_1}^{s_1})_{\vartheta, q^*}$ avec $s^* = (1-\vartheta)s_0 + \vartheta s_1$
$H_{p^*}^s = (H_{p_0}^s, H_{p_1}^s)_{\vartheta, p^*}$ $\quad \frac{1}{p^*} = \frac{1-\vartheta}{p_0} + \frac{\vartheta}{p_1}$

et $B_{p,q}^s \subset H_p^s \subset B_{p,\infty}^s$. On a $H_p^s \hookrightarrow L_{p_1,\infty}$
$H_p^s \subset B_{p,\infty}^s = (B_{p,1}^{s_1}, B_{p,1}^{s_2})_{\theta,\infty}$ $\quad s_i - \frac{1}{p} = 0 - \frac{1}{p_i}$
$\subseteq (L_{p_1}, L_{p_2})_{\theta,\infty}$.
$(H_{p_1}^s, H_{p_2}^s)_{\eta,p}$