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Uniform convex renormings

Uniform convexity:

modulus: $\delta_X(\varepsilon) = \inf\limits_{\substack{\|x-y\|\geq\varepsilon \\ \|x\|,\|y\|=1}} 1 - \left\|\frac{x+y}{2}\right\|$

· $\delta_X(\varepsilon) \leq \delta_{\mathbb{R}^2}(\varepsilon) = 1-\left(1-\frac{\varepsilon^2}{4}\right)^{1/2} \sim \frac{\varepsilon^2}{8}$

· X is UC if $\delta_X(\varepsilon)>0$ for $\varepsilon>0$.

· A Banach space admits an equivalent uniformly convex norm iff X is superreflexive iff its unit ball does not contain ε-trees of arbitrary height.
(James, Enflo, Pisier) (Pisier added that one can get $\delta(\varepsilon)\geq c\,\varepsilon^p$ for some $c>0$, $p\geq 2$)

New idea: from Gilles Lancien's PhD thesis: slice derivation, dentability index

$d'_\varepsilon(A) = A \setminus \bigcup\{H \text{ open halfspace} : \operatorname{diam} A\cap H < \varepsilon\}$.

$Dz(A,\varepsilon) = \inf\{n : d^n_\varepsilon(A)=\emptyset\}$...

If $Dz(A,\varepsilon)$ is finite for every $\varepsilon>0$, A is said to be finitely dentable.

X is UC $\Leftrightarrow$ $\forall\varepsilon>0\ \exists\vartheta>0$ $d'_\varepsilon(B_X) \subset (1-\vartheta)B_X$.

Thus B_X is finitely dentable. One even has $Dz(B_X,\varepsilon) \leq \delta_X(\varepsilon)^{-1}$.

If $|||\cdot|||$ is 2-$\simeq$ to $\|\cdot\|$, $Dz(B_{|||\cdot|||},\varepsilon)\leq \delta_{|||\cdot|||}(\varepsilon)^{-1}$

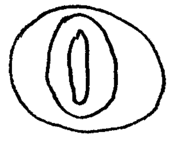
$Dz(B_X,4\varepsilon)^{-1} \leq$

Thus $\Delta(\varepsilon) = Dz(B_X,\varepsilon)^{-1}$ is an upper bound of all the modulis of convexity of 2-equiv. norms. But is it the best? How can one use Dz to renorm?

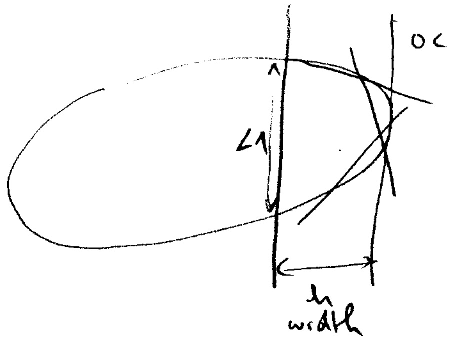
are those necessary?

G. Lancien: If $Dz(B_X)\leq\omega$, then $\delta_X(\varepsilon) \geq \dfrac{c\,\varepsilon^2}{Dz(B_X,\varepsilon/8)^{2}}$ for some $\simeq$ norm.

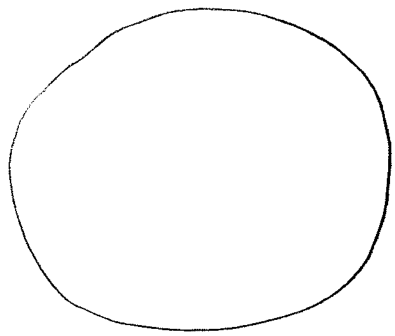
The exponent is very difficult to remove: lack of isotropy!



I gave up and took up another index:



you remove slice not intersecting the rest of the convex body



Th : X superreflexive, $\varepsilon \in]0,1]$. There is $N_\varepsilon \in \mathbb{N}$ s.t. for every slice $S = A \cap H$ with A convex, H halfspace, of diam ≤ 1, of width $h>0$, there are cl. cvx sets $(C_n)_{n=0}^{N_\varepsilon}$ s.t.

a) $A = C_0 \supset \dots \supset C_{N_\varepsilon} \supset A \setminus H$

b) $d'_\varepsilon(C_{n-1}) \subset C_n$

c) $C_{N_\varepsilon} \cap H$ has width $\leq \frac{h}{2}$.

This defines the index $\mathfrak{M}_X(\varepsilon)$. The least N_ε.

- $\gamma \in]0, \frac{1}{10}[$: If X is superreflexive, there is (for every $\varepsilon>0$) $|||\cdot|||$ equivalent s.t. B_X

$$1 - |||\frac{x+y}{2}||| > \frac{\gamma}{20\,\mathfrak{M}_X(\varepsilon)} \quad \text{if } |||x||| = |||y||| = 1 \text{ and } \|x-y\| > 16\varepsilon.$$

$$\delta_{|||\cdot|||}(\varepsilon) \geq \frac{c}{|\log \varepsilon|\,\mathfrak{M}_X(\varepsilon)}$$

- If X is superreflexive, there is $c>0$ such that $\mathfrak{M}_X(\varepsilon) \leq c\, \delta_{|||\cdot|||}(\varepsilon)^{-1}$ for any 2-$\simeq$ norm $|||\cdot|||$ on X.

put the slice into a cylinder

Sketch of proof: 1) treat slices of 2-eq. balls.

2) add $\eta \|\cdot\|_{UC}$ to make $|||\cdot|||$ uniformly convex, etc.

this also suggests another index $\mathfrak{N}_X$ which is submultiplicative.

Th there is a positive decreasing submultiplicative $\mathfrak{N}_X$ s.t. $\mathfrak{N}_X^{-1}$ is the supremum wrt $\precsim$ ($\leq$ up to a constant) of $\{\delta_{|||\cdot|||}(t) : |||\cdot||| \text{ eq. norm on } X\}$

Q. is the sup ~~of~~ a maximum?

Some consequences.

- $\eta_X(\varepsilon)$ is an isometric invariant of X.
- $Dz(B_X, \varepsilon) \leq \eta_X(\varepsilon) \lesssim \varepsilon^{-2} Dz(B_X, \varepsilon)^2$
- If $Dz(B_X, \varepsilon) < \omega$ for $\varepsilon \in]0, \frac{1}{2}[$, then X is superreflexive.

  (James showed that it suffices that $\delta_X(t) > 0$ for some $t \in]0,2[$)

- These methods also apply to the Kuratowski index: (Kuratowski measure of noncompactness)
- There may be a way to consider also infinite ordinals.

  (But are the bounds interesting? $Dz(C) < \omega^{Sz(C)}$?)