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Box spaces and coarse geometry:

$X \sim_{CE} Y$ if $\exists f : X \hookrightarrow Y$ coarse embedding $\exists c > 0$ $\forall y \in Y$ $d(\mathrm{Im} f, y) < c$

N.B.: $X \hookrightarrow \mathcal{H}$ (Hilbert space) implies the coarse Baum Connes conjecture for X. (Yu 2000)

If $X = \mathrm{Cay}(G, S) \hookrightarrow \mathcal{H}$, you have the Novikov conjecture (Yu).

Def: (Yu 2000) (X, d_X) with bounded geometry has property A if $\forall R > 0$ $\forall \varepsilon > 0$ $\exists S > 0$ $\exists \varphi : X \to \ell^2(X)$

$\|\varphi(x)\|_2 = 1$ and $d(x,y) < R \Rightarrow \|\varphi(x) - \varphi(y)\|_2 < \varepsilon$ and $\mathrm{supp}\, \varphi(x) \subset B(x, S)$

(= non equivariant notion of amenability)

N.B.: If X has A, then $X \hookrightarrow \mathcal{H}$.

- What doesn't have property A? ... box spaces!

Def G f.g. is <u>residually finite</u> if $\forall g \in G \setminus \{e\}$ $\exists \pi : G \twoheadrightarrow F$ finite

$\pi(g) \neq e \in F$ $\quad (\Leftrightarrow \bigcap_{N \trianglelefteq G \text{ finite index}} N = 1.$

Consider $G = \langle S \rangle$, S finite.

Def: Let $\{N_i\}$ be a sequence of finite-index nested normal subgroups

$G \trianglerighteq N_1 \trianglerighteq N_2 \trianglerighteq \dots$ s.t. $\bigcap N_i = 1$. The box space $\square_{\{N_i\}} G$ of G wrt $\{N_i\}$

is $\bigsqcup_i G/N_i$ with metric d s.t. $\left| \begin{array}{l} d|_{G/N_i} = \text{induced Cayley graph metric.} \\ d(G/N_i, G/N_j) \geq \max\{\mathrm{diam}\, \frac{G}{N_i}, \mathrm{diam}\, \frac{G}{N_j}\} \end{array} \right.$

Property: $\square_{\{N_i\}} G$ has prop A $\Leftrightarrow$ G amenable.

Q: Is $A \Leftrightarrow \hookrightarrow \mathcal{H}$?

A: No (Nowak 2007) for any finite F, $\bigsqcup_i \bigoplus^i F \hookrightarrow \mathcal{H}$

does not have A

Pb: It doesn't have bounded geometry either.

A: No for bounded geometry (Arzhantseva - G - S 2011) by a box space!

Counter-Example $\mathbb{F}_2$ = free group, $N_1 = \mathbb{F}_2$, $N_{i+1} = N_i^2 = \langle g^2 \mid g \in N_i \rangle$

Then $N_i / N_{i+1} = \overset{r_i}{\bigoplus} \mathbb{Z}_2$

Main idea: • $\mathbb{F}_2$ nonamenable.

• each quotient is a covering space of the previous one.

$$\begin{array}{c} \mathbb{F}_2/N_{i+1} \\ \downarrow \\ \mathbb{F}_2/N_i \end{array}$$

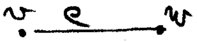
Covering space: $\tilde{X}$ corresponding to $\rho : \pi_1(X) \twoheadrightarrow K$

$\pi \downarrow$ $\quad \tilde{X}$ has edges $E(\tilde{X}) = E(X) \times K$

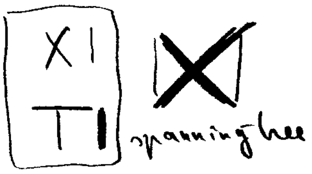
$T \supset X$ $\quad$ vertices $V(\tilde{X}) = V(X) \times K$

spanning tree [the edges not in T correspond to the generators of $\pi_1(X)$]

Let us specify the edges: (e,h) joins $(v,h),(w,h)$ if $e \in T$.

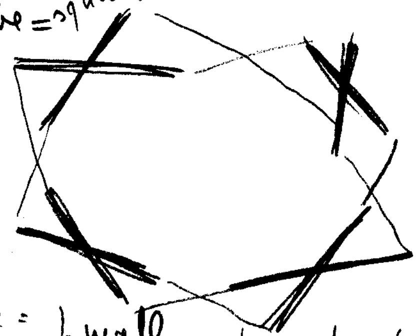


(e,h) joins (v,h) and $(w, \rho(e)h)$ if $e \notin T$



$K = \mathbb{Z}_2 \times \mathbb{Z}_2$: 4 copies of spanning trees = 4 clouds.

This looks like the 2 dim cube = square.



We get a wall metric on the covering space.

wall structure : if you remove a wall, it leaves behind 2 connected components. structure = {walls s.t. each edge is in at least one wall}.

Here walls are $\{\pi^{-1}(e) : e \in X\}$. We have a wall metric d_W.

This yields a coarse embedding of $(\square_{\{N_i\}} \mathbb{F}_2, d_W) \hookrightarrow \mathcal{H}$.

And $d_W \sim d$ + girth argument shows $(\square_{\{N_i\}} \mathbb{F}_2, d) \hookrightarrow \mathcal{H}$.

Def: Let $m \geq 2$ be an integer. The derived m-series of G is $N_1 = G$,

$N_{i+1} = N_i^m [N_i, N_i]$ $\quad (N_i / N_{i+1} = \bigoplus \mathbb{Z}_m)$

Theorem: Given a f.g. free group $\mathbb{F}_n$, the box space $\square_{\{N_i\}} \mathbb{F}_n$ wrt the derived m-series N_i embeds into $\mathcal{H}$.

Idea: $\mathbb{F}_n/N_{i+1}$ $\downarrow$ $\mathbb{F}_n/N_i$ $\quad K = \oplus \mathbb{Z}_m \qquad \sqcup \oplus \mathbb{Z}_m \hookrightarrow \mathcal{H}$ ⚠

Define a metric d_Q s.t. $(\sqcup \mathbb{F}_n, d_Q) \hookrightarrow \mathcal{H}$ and $d_Q \sim d$.

Notation: For $X \in F_i/N_{i+1}$ and T spanning tree and C^T_X the cloud containing T s.t. $X \in$ cloud; $d_T = \oplus\mathbb{Z}_m$-metric between clouds;

$N_e = \#$ spanning trees $\not\ni e = N$

Let $d_Q(x,y) = \frac{1}{N} \sum_T d_T(C^T_x, C^T_y)$: with this, we can embed into Hilbert space.

Q: What about compression with exponent $\geqslant \frac{1}{2}$: Kaminker → implies property A.

Here it is again $\frac{1}{2}$.