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Structure theory of algebras of type III vN algebras.

solidity

Def: a vN algebra is a unital $*$-subalgebra A of $B(H)$ s_{op}-closed.

Bicommutant theorem: $A \subset B(H)$ unital $*$ algebra.
A vN $\Leftrightarrow$ $A = A'' = \{S \in B(H) \mid \forall T \in A \; ST = TS\}$

Def: a factor is a (topologically) simple vN algebra $M \Leftrightarrow Z(M) = \mathbb{C}1$. ($M'' \cap M'$)
If one wants to generalize representation theory from finite to infinite groups, the right notion to study is: a finite factor, that is, (M, τ) with $\tau : M \xrightarrow{\text{functional}} \mathbb{C}$ with $\tau(yx) = \tau(xy)$

Th: (Bekka)$^{2004}_{n \geq 3}$ Let $\pi : SL(n, \mathbb{Z}) \to \mathcal{U}(M)$ with $M = \pi(SL(n,\mathbb{Z}))''$ a finite factor. Then either $\dim_{\mathbb{C}} M < \infty$ or $M \cong L(SL(n, \mathbb{Z}))$
where $L(\Gamma)$ is the group vN algebra.

Solidity: A finite factor M is called prime if
$M = M_1 \bar{\otimes} M_2$ implies $\dim_{\mathbb{C}} M_i < \infty$ for $i = 1$ or $i = 2$.
th (Ge 1997) the $L\mathbb{F}_n$, $n \geq 2$ are prime.

Ozawa$_{2004}$ introduced solidity: M finite vNa is solid if
$A \subseteq M$ diffuse implies $A' \cap M$ amenable.
Here: N vNa is diffuse iff for p nonzero projection there is $q \leq p$ nonzero projection s.t. $q \neq p$ and $qp = q$.
(ex. $N = L^\infty(X)$ is diffuse $\Leftrightarrow$ X nonatomic)
Recall that Connes proved M amenable $\Leftrightarrow$ M hyperfinite (Connes)
and M is hyperfinite if $M = \overline{\bigcup_{i \in \mathbb{N}} M_i}$, M_i finite-dimensional vN algebras.

Proposition: M solid finite vN algebra implies M is either prime or amenable.
Proof: Assume M not prime but solid: let $M = M_1 \bar{\otimes} M_2$, M_1, M_2 ∞-dim.
We shall cheat and assume the M_i's are factors. Then they are diffuse.

But $M_1 \otimes 1$ diffuse implies $1 \otimes M_2 = (M_1 \otimes 1)' \cap M$ hyperfinite
Also $M_1 \otimes 1$ is hyperfinite, so that $M = M_1 \bar{\otimes} M_2$ is hyperfinite

Solidity has nice stability properties

Ozawa$_{2004}$: Γ hyperbolic $\Rightarrow L\Gamma$ is solid.

Today: strong solidity is studied: cf Vaes, Popa

Focus on work with Cyril Houdayer.

- Solidity for general vN algebras: M is solid if $\forall A \subseteq M$ diffuse with $E : M \longrightarrow A$ normal conditional expectation $A' \cap M$ is amenable.
- Ultrapowers (Ocneanu, Ando - Haagerup) : M finite factor algebra with trace τ, ω ultrafilter on $\mathbb{N}$: then $M^\omega = \dfrac{\ell^\infty(\mathbb{N}, M)}{\{\tau(x_n^* x_n) \xrightarrow{n \to \omega} 0\}}$ is a v. N algebra.

Here we cheat a little bit: one must consider a lesser space than $\ell^\infty(\mathbb{N}, M)$ in the type III_1 case.

- ω - solidity. M vNa is ω-solid if for all $A \subseteq M$ without amenable direct summand but with $E : M \longrightarrow A$ conditional expectation, $A' \cap M^\omega$ is discrete.

. Consequence : $A' \cap M^\omega = A' \cap M$.

Note: Ozawa concluded solidity of $L\Gamma$ from an AO = Akemann - Ostrand property. [ex. hyperbolic groups, lattices in $SO(1,n)$.

Theorem: (Houdayer - Raum) : Let M be a vNa with (AO), then M is ω-solid. In particular, the free Araki-Wood factors are ω-solid

Application: M ω-solid vNa with faithful normal state $\varphi \in M_*$ such that M^φ is a nonamenable factor. Then $\tau(M)$ is the smallest topology on $\mathbb{R}$ which makes $t \longmapsto \sigma_t^\varphi$ continuous.

$\hookrightarrow$ if you have a state φ on M_*, you can introduce $\sigma_t^\varphi \in \mathrm{Aut}(M)$ st. $\varphi(xy) = \varphi(y \sigma_{i}^\varphi(x))$

$\tau(M)$ is def'd st $t_n \longrightarrow 0 \Leftrightarrow \exists (u_n) \subset \mathcal{U}(M)$ $\mathrm{Ad}\, u_n \circ \sigma_{t_n}^\varphi \longrightarrow \mathrm{Id}$.