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Compact Quantum semigroups.

Def: Let (A, Δ) be: – A unital C^*-algebra
– $\Delta : A \to A \otimes A$ unital $*$ homomorphism that is coassociative: $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$ is "comultiplication"

Idea: if P is a compact semigroup with $(x,y) \mapsto x \cdot y$ and $\mathcal{C}(P)$ the space of continuous functions, then

$$\Delta : \mathcal{C}(P) \to \mathcal{C}(P) \otimes \mathcal{C}(P) \cong \mathcal{C}(P \times P)$$
$$f \mapsto ((x,y) \mapsto f(x \cdot y))$$

We identify $(\mathcal{C}(P), \Delta)$ with P.

Conversely, if (A, Δ), A commutative C^*-algebra, then $A \cong \mathcal{C}(P)$, P compact. Δ is always from on A
And $x \cdot y$ is defined via $f(x \cdot y) = \Delta(f)(x,y)$. We say $(A, \Delta) \cong P$.

We say (A, Δ) is compact q group if $\{(a \otimes I) \cdot \Delta(b) : a, b \in A\}$ and $\{(I \otimes a) \cdot \Delta(b) : a, b \in A\}$ are dense in $A \otimes A$

Let S be a cancellative Abelian semigroup $(S, +, 0)$ and Γ be the Grothendieck group of S: $\Gamma = \{b - a : a, b \in S\}$

Then ~~let~~ $H = \ell^2(S)$ with basis $\{e_a\}_{a \in S}$ and $e_a(b) = \delta_{a,b}$ for $a, b \in S$.

For $a \in S$, let $T_a \in B(H)$ and $T_a e_b = e_{a+b}$ for $b \in S$.

Then $a \mapsto T_a$ is a regular expression: $T_a^* e_b = \begin{cases} e_c, & b = a + c, c \in S \\ 0 \end{cases}$

then the T_a, T_b^*, $a, b \in S$ generate $C_r^*(S)$, the reduced semigroup C^*-algebra.

Def: G. Murphy, 1987, J. Operator th.

Example: $S = (\mathbb{Z}_+, +)$, $\Gamma = \mathbb{Z}$, $C_r^*(S) = \mathcal{T}$, the Toeplitz algebra.

Example: $S = \{0, 2, 3, 4, 5, ...\}$ or $\{0, 3, 6, 7, 8, ...\}$ "perforated semigroup"
(then Γ is still $= \mathbb{Z}$)

Th (Coburn) All isometric irreducible faithful rep. are $S \xrightarrow{\pi_1} C_{\pi_1}^*(S)$, $S \xrightarrow{\pi_2} C_{\pi_2}^*(S)$, $C_{\pi_1}^*(S) \xrightarrow{\varphi} C_{\pi_2}^*(S)$

We define the natural order on S by $a \leq b \iff \exists c \in S \; a + c = b$.

R. G. Douglas and Murphy proved that if the natural order is total, then there is a canonical isomorphism.

The monomials in $C^*_r(S)$ are the finite products of T_a and T_b^*.

Thm: For any monomial V in $C^*_r(S)$, there is $a, b \in S$ with $\lim_{c\in S} T_c^* V T_c = T_a^* T_b$.

i.e., $\exists c \, \forall b \geq c \; T^*_d U T_d = T_a^* T_b$.

Define $\operatorname{ind}[ex] V = b - a \in \Gamma$. Then $\operatorname{ind}(VW) = \operatorname{ind} V + \operatorname{ind} W$

$\operatorname{ind} V^* = -\operatorname{ind} V$.

Denote by $\mathcal{A}_c$ the space generated by the monomials with index c

then 1) $\mathcal{A}_c \cdot \mathcal{A}_b \subseteq \mathcal{A}_{c+b}$

2) $\mathcal{A}_0$ is a commutative C^*-algebra.

$$V e_a = \begin{cases} e_{a+\operatorname{ind} V} \\ 0 \end{cases}$$

example: if $S = \{0, 2, 3, 4, \dots\}$, then $T_2 T_2^*$: $e_0 = 0$;

$T_3 T_2 T_2^* T_3^*$: $e_0 = 0$, $e_2 = 0$.

$\operatorname{ind}(T_a^* T_b T_c^* T_d) = b + d - a - c$.

The V's are in fact commutative C^*-algebras.

3) If $c = b - a$, $a, b \in S$, then $\mathcal{A}_c = T_a^* \mathcal{A}_0 T_b$.

Assume $V \in \mathcal{A}_c$: $\operatorname{ind} V = c$. Then $T_a^* \underbrace{[T_a V T_b^*]}_{\text{index } 0} T_b = V$

If $a \in S$, consider $P_a : \ell^2(S) \to \mathbb{C} e_a$: it is a projection and

for $c \in \Gamma$, $Q_c : C^*_r(S) \to C^*_r(S)$

$$Q_c(A) = \bigoplus_{\substack{a\in S\\ a+c\in S}} P_{a+c} A P_a$$

Th $Q_c : C^*_r(S) \to \mathcal{A}_c$:

Proof: If $A \in C^*_r(S)$, $A = 0 \iff Q_c(A) = 0$ for all $c \in \Gamma$.

And Q_0 is the conditional expectation of $C^*_r(S)$ on $\mathcal{A}_0$.

$C^*_r(S) = \bigoplus_{c\in\Gamma} \mathcal{A}_c$ if $A \in C^*_r(S)$, then $A = \bigoplus_{c\in\Gamma} Q_c(A)$; the algebra is graded.

If K is the commutator ideal in $C^*_r(S)$, $K = \{AB - BA \mid A, B \in C^*_r(S)\}$

Th: $A \in K \iff \lim_{c\in S} T_c^* A T_c = 0$.

then $C^*_r(S)/K \cong C(G)$, G a compact group dual to Γ.

Example: if $S = \mathbb{Z}_+$, $\mathcal{T}/K \cong \mathcal{C}(S^1)$
unit circle

Consider $\mathcal{C}(G)$, χ^a, $a \in \Gamma$: $\tau : G \to \operatorname{Aut}(C^*_r(S))$

$\tau(\alpha) T_a = \chi^a(\alpha) T_a$: $\tau(\alpha) T_a^* = \chi^{-a}(\alpha) T_a^*$

τ is a representation of G in $\operatorname{Aut}(C^*_r(S))$.

We get $(C^*_r(S), G, \tau)$, $(K, G, \tau|_K)$

If we have $(C^*_r(S), \Delta)$ a compact semigroup and $\Delta: C^*_r(S) \to C^*_r(S) \otimes C^*_r(S)$
$\Delta(f)(x,y) = f(x\cdot y)$

and for $\chi^a, a \in G$: $\Delta(\chi^a)(x,y) = \chi^a(x\cdot y) = \chi^a(x)\chi^a(y) = (\chi^a \otimes \chi^a)(x,y)$:
$\Delta(\chi^a) = \chi^a \otimes \chi^a$.

If $H = L^2(G)$, $f \in C(G)$, $L_f \in B(H)$: $L_f g = f \cdot g$ $\quad (C(G) \to B(H))$.

We let $L^2(G) = \langle \chi^a : a \in \Gamma \rangle$ and $H_S = \langle \chi^a : a \in S \rangle$.

$P_S : L^2(G) \to H_S$ projection, $T_f = P_S L_f P_S$.

If $f \in C(G)$, $T_f \to C^*_r(S)$ and $\Delta(T_a) = T_a \otimes T_a$, $\Delta(V) = V \otimes V$.

<u>Th</u>. $(C^*_r(S), \Delta)$ is a compact q semigroup: we have $\pi: C^*_r(S) \to K$

$G \cong (C(G), \Delta_G) \subseteq (C^*_r(S), \Delta)$

$$\begin{tikzcd} C^*_r(S) \arrow[r] \arrow[d, "\pi"'] & C^*_r(S) \otimes C^*_r(S) \arrow[d, "\pi \otimes \pi"] \\ C(G) \arrow[r, "\Delta_G"] & C(G) \otimes C(G) \end{tikzcd}$$

$\mathcal{A}^*$ has a Banach algebra structure,

$\mathcal{A} = C^*_r(S)$, $(\varphi * \psi)(A) = (\varphi \otimes \psi)\Delta(A)$, $A \in \mathcal{A}$.

the Haar functional $h \in \mathcal{A}^*$: $h * \varphi = \varphi * h = \lambda_\varphi \cdot h$. for $\varphi \in \mathcal{A}^*$ $\lambda_\varphi \in \mathbb{C}$.