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How noncommutative is the noncommutative topological entropy?

Notion introduced by Voiculescu. Connection with comm. entropy?

Let (X, d) be a compact metric space and $T: X \xrightarrow{\text{continuous}} X$

How chaotic is T? How does it move things around.

Kolmogorov Sinai: measurable entropy : This is difficult to put in a nc framework, but this is easily shown to generalise -
MacAndrew Konheim: topological entropy

Richard Bowen (1970s): $\varepsilon > 0$, $k \in \mathbb{N}$. Say $F \subset\subset X$ is (k, ε)-spanning if

$\forall_{x \in X} \exists_{f \in F} \forall_{i=0}^{k-1} d(T^i x, T^i f) < \varepsilon$. "$\varepsilon$-nets on the orbits"

Let $s(n, \varepsilon)$ be $\min\{\# F : F \ (n, \varepsilon)\text{-spanning}\}$

Take $h_{top} = \sup_{\varepsilon > 0} \overline{\lim_n} \frac{1}{n} \log s(n, \varepsilon)$.

Does not depend on the equivalent metric.

Consider $C_N = \prod_{i=1}^{\infty} \{1, \dots, N\}$ "words in the alphabet $\{1, \dots, N\}$".

Consider $\sigma: C_N \to C_N$, $\omega \mapsto (\omega_{\cdot+1})$ the Bernoulli shift: then $h_{top}(\sigma) = \log N$.

Ornstein - Weiss: entropy characterises the Bernoulli shift.

<u>Voiculescu's entropy</u>: A unital nuclear C^*-algebra, $\alpha: A \to A$ unital *-hom. (If $A = C(X)$, $\alpha_T(f) = f \circ T$)

$$A \xrightarrow{id} A, \quad A \to M_n \to A$$

Let $\Omega \subset\subset_{\text{finite}} A$, $\varepsilon > 0$:

$CPA(\Omega, \varepsilon) = \{(\varphi, \psi, M_n) : \psi: M_n \to A, \ \varphi: A \to M_n, \ \varphi, \psi \text{ c.p. unital}: \forall_{a \in \Omega} \|a - (\psi \circ \varphi)(a)\| < \varepsilon\}$

is nonempty by nuclearity.

Question what if you factor only by ℓ^∞_n?

Let $ncp(\Omega) = \min\{n : (\varphi, \psi, M_n) \text{ in } CPA(\Omega, \varepsilon)\}$.

$ht(\alpha) = \sup_{\substack{\varepsilon > 0 \\ \Omega \subset\subset A}} \overline{\lim} \frac{1}{n} \log ncp(\Omega^{(n)}, \varepsilon)$ where $\Omega^{(n)} = \bigcup_{k=0}^{n-1} \alpha^k(\Omega)$

Properties: (i) $ht(\alpha_T) = h_{top} T$! Thus we generalise the commutative concept.

But this is difficult!

Idea: $ht\, \alpha_T \le h_{top}(T)$: easy: approximations, diagonal matrices.

$ht\, \alpha_T \geq h_{top} T$ is very difficult. how to produce finite sets?

Voiculescu proves this with vN algebra techniques, measure entropy (CNT)

+ one-sided variational principle + approximation
→ "asymptotic commutativity" ... sup of metric entropy ...

(ii) Let $B \subset A$, B C^*-alg., and $\alpha(B) \subset B$. Then $ht\,\alpha|_B \leq ht\,\alpha$

Problem: B need not be nuclear; but it is exact ... (Brown) ...

and all works for exact C^*-algebras ...

How to compute $ht\,\alpha$? (i) Find some approximations to get $ht\,\alpha \leq M$.

(ii) Find $B \subset A$, B invariant under α, prove $ht\,\alpha|_B \geq M$

This works in particular if B is commutative.

Example: Cuntz algebras: Let $O_N = C^*\{S_1, \dots, S_N$ isometries: $S_i^* S_j = \delta_{ij} 1$, $\sum S_i S_i^* = 1\}$

Let $\Phi \in \mathrm{End}(O_N)$, $\Phi(x) = \sum_{i=1}^{N} S_i x S_i^*$ for $x \in O_N$. $I = \bigcup_{k \in \mathbb{N}} \{1, \dots, N\}^k$

O_N has 2 subalgebras: $F_N = \overline{\mathrm{lin}}\, \{S_\mu S_\nu^* : \mu, \nu \in I, |\mu| = |\nu|\}$.

(here $S_\mu = S_{\mu_1} \cdots S_{\mu_{|\mu|}}$)

$\cong UHF(N^\infty) = \bigotimes_{i=1}^{\infty} M_N$.

$C_N = \overline{\mathrm{lin}}\, \{S_\mu S_\mu^* : \mu \in I\} \cong C(C_N)$. Marie Choda.

We have $\phi(C_N) \subset C_N$, $\phi|_{C_N} \cong \alpha_\sigma$. Then $ht\,\phi \geq \log N$. !

Actually =

Let us now define the commutative version

$ht_c(\alpha) = \sup\{ht\,\alpha|_B : B \subset A, B$ commutative, $\alpha(B) \subset B\}$.

Q: Do we have $ht_c(\alpha) = ht(\alpha)$? NO!

(1) Take $A = K(H) \oplus \mathbb{C}1$, U a unitary on H with purely singular spectrum.

Then $\alpha_U(x) = U x U^*$ has only $\mathbb{C}1$ as comm. inv. C^*-subalgebra.

But it turns out that $ht\,\alpha_U = 0$!

(2) Consider O_2 and consider $\rho(S_1) = S_1 S_2 S_1^* + S_1 S_1 S_2^*$.
$\rho(S_2) = S_2$

Then $\rho(C_N) \subset C_N$ and $ht\,\rho|_{C_N} = 0$ and $ht\,\rho = \log 2$

But it turns out that there is a comm. sub. C_2' of O_2 s.t.

$\rho|_{C_2'} = \alpha_\sigma$.

Take 3: Consider Powers' algebra of bitstream shifts.

Let $Z \subset \mathbb{N}$, $A_Z = C^*\{u_i : i \in Z\}$. $u_i = u_i^*$, $u_i^2 = 1$, $u_i u_j = (-1)^{\chi_Z(|i-j|)} u_j u_i$
$\underbrace{\qquad\qquad}_{\text{symmetries}}$

Consider $\alpha \in \mathrm{Aut}\, A_Z$, $\alpha(u_i) = u_{i+1}$

Theorem: Suppose that Z is "wild enough" such that have full measure in $\{0,1\}^{\mathbb{N}}$.
Then $ht_c(\alpha) = 0 < \frac{\log 2}{2} \leq ht(\alpha) \leq \log 2$

You can use the picture $C^*(\prod_{i=-\infty}^{+\infty} \mathbb{Z}_2, \sigma)$

Idea: $\alpha \otimes \alpha$ ... commutation relation.

Let's go back to Cuntz algebras: O_N:

Endomorphisms of O_N: combinatorial object

Takesaki - Cuntz: there is a 1-1 correspondence between unitaries in O_N and endomorphisms of O_N: $U \in \mathcal{U}(O_N)$: $\rho_U(S_i) = U S_i$.

Conversely, if $\rho \in \mathrm{End}(O_N)$, $U_\rho = \sum_{i=1}^{N} \rho(S_i) S_i^*$

ex: shift endomorphism: $\Phi \longmapsto U_\Phi = \sum_{i,j=1}^{N} S_i S_j S_i^* S_j^*$ (Kawamura)

$J_k = \{1, \dots, N\}^k$, $\pi \in \mathrm{Perm}(J_k)$: $U_\pi = \sum_{u \in J_k} S_u S_{\pi(u)}^* \in \mathcal{B}_N$

If $U \in \mathcal{F}_N$, then $\rho_U(\mathcal{F}_N) \subset \mathcal{F}_N$.

If $U \in \mathcal{B}_N$, then $\rho_U(\mathcal{B}_N) \subset \mathcal{B}_N$.

Th (w. Zacharias): If $U \in \mathcal{F}_N^k = \lim \{S_\mu S_\nu^* : |\mu| = |\nu| \leq k\}$, then $ht\, \rho_U \leq (k-1) \log N$.

Another counterexample to the "commutative question" that is irreducible

Th. Let $V \in \mathcal{B}(\mathbb{C}^N \otimes \mathbb{C}^N)$ a multiplicative unitary and $V_{12} V_{13} V_{23} = V_{23} V_{12}$

let F = flip on $\mathbb{C}^N \otimes \mathbb{C}^N$: $VF \in M_N \otimes M_N \subset \mathcal{F}_N \subset O_N$ (in $\mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N$)

Then $ht\, \rho_{VF} = \log N$

"shift on the tower of subfactors"

cook up some other examples: what of $k = 1$?

$U \in \mathrm{lin}\{S_i S_j^* ; i,j = 1 \dots N\} \cong M_N$ and ρ_U is just changing labels "globally", a so-called Bogolyubov automorphism: $ht\, \rho_U = 0$. However, you can find $\rho \in \mathrm{End}(O_N)$ s.t. $ht\, \rho = 0$, $ht\, \rho \circ B = \frac{1}{2} \log 2$

Bogolyubov automorphism

Q: subhomogeneous C^*-algebras: bounded blocks ... factorisations...

Expert: David Kerr: there cannot be a "commutative proof"

Open Q: automorph of hyperfinite II_1 factor: Is CNT entropy "commutative" in this case?