M. Kraus 18/11/2014

Def.: (M, d_M), (N, d_N) metric spaces, $T: M \to N$ is a coarse embedding if $\exists \rho_1, \rho_2 : [0,\infty) \to [0,\infty)$ non-decreasing s.t.

* $\lim_{t \to \infty} \rho_1(t) = \infty$
* $\rho_1(d_M(x,y)) \le d_N(T(x), T(y)) \le \rho_2(d_M(x,y)) \; \forall x, y \in M$

Notation: $M \underset{c}{\hookrightarrow} N$

Thm (Randrianarivony): X quasi Banach space. Then

$X \underset{c}{\hookrightarrow}$ Hilbert space $\iff X \hookrightarrow L_0(\mu)$ for some proba space $(\Omega, \mathcal{F}, \mu)$

lin. embeds — with topology of CV in probability

Problem: X quasi-Banach space, $X \underset{c}{\hookrightarrow}$ Hilbert. How well can we coarsely embed X into a Hilbert?

Def: (Guentner, Kaminker): M, N metric spaces, M unbounded

- $T: M \to N$ is large-scale Lipschitz if $\exists A > 0 \; \exists B > 0$

  s.t. $d_N(T(x), T(y)) \le A\, d_M(x,y) + B \quad \forall x, y \in M$
- compression exponent of M in N

  $$\alpha_N(M) = \sup\{\alpha \ge 0 : \exists \text{ large-scale Lip. } T: M \to N \; \exists\, C, \epsilon > 0 \text{ s.t. } d_N(T(x), T(y)) \ge C d_M(x,y)^\alpha \text{ if } d_M(x,y) \ge \epsilon\}$$
- Hilbert space compression exponent of M

  $$\alpha(M) = \sup\{\alpha \ge 0 : \exists \text{ large-scale map } T: M \to H \; \exists\, C, \epsilon > 0 \text{ s.t. } \|T(x) - T(y)\|_H \ge C d_M(x,y)^\alpha \text{ if } d(x,y) \ge \epsilon\}$$

Facts ∘ $\alpha_N(M) \leq 1$, $\alpha(M) \leq 1$ (M unbounded)

∘ $\alpha_N(M) > 0 \Rightarrow M \hookrightarrow N$

$\alpha(M) > 0 \Rightarrow M \hookrightarrow$ Hilbert

the closer $\alpha_N(M)$ ($\alpha(M)$) is to 1, the better $M \hookrightarrow N$ ($M \hookrightarrow H$)

Rem.: ∘ Γ finitely generated group with a word length metric

∘ Γ exact $\Rightarrow \Gamma \hookrightarrow$ Hilbert (Guentner, Kaminker, Ozawa)

∘ $\alpha(\Gamma) > \frac{1}{2} \Rightarrow \Gamma$ exact (Guentner, Kaminker)

Def.: ∘ (M, d_M) metric space, $\alpha \in (0,1) \Rightarrow$ ~~the~~ d_M^α is also a metric

(M, d_M^α) is α-snowflaking of (M, d_M)

∘ (N, d_N) another metric space

$$\Rightarrow S_N(M) = \sup\{\alpha \in (0,1] : (M, d_M^\alpha) \underset{L}{\hookrightarrow} (N, d_N)\}$$

$$S(M) = \sup\{\alpha \in (0,1] : (M, d_M^\alpha) \underset{L}{\hookrightarrow} \text{Hilbert}\}$$

Facts: M unbounded $\Rightarrow 0 \leq S_N(M) \leq \alpha_N(M) \leq 1$

Recall: ∘ $0 < p < \infty$: $L_p(\mu)$ equipped with a (quasi-)norm

$$\|f\|_p = \left(\int |f|^p d\mu\right)^{\frac{1}{p}}$$

∘ $0 < p < 1$: the canonical metric on $L_p(\mu)$ is

$$d_p(f,g) = \|f-g\|_p^p = \int |f-g|^p d\mu$$

Albiac-Baudier: $0 < p \leq q < \infty \Rightarrow S_{\ell_q}(\ell_p) = \alpha_{\ell_q}(\ell_p) = \dfrac{\max\{p,1\}}{\max\{q,1\}}$

In particular,

$$0 < p \leq 2 \Rightarrow s(\ell_p) = \alpha(\ell_p) = \frac{\max\{p, 1\}}{2}$$

$$p > 2 \xRightarrow{\text{Johnson-Bendivinarivsky}} \ell_p \not\hookrightarrow_c \text{Hilbert} \Rightarrow s(\ell_p) = \alpha(\ell_p) = 0$$

X quasi Banach space

Aoki, Rolewicz

$\Rightarrow \exists p \in (0,1]$ $\exists$ equivalent quasi norm $\|\cdot\|$ on X

s.t. $\|x+y\|^p \leq \|x\|^p + \|y\|^p \quad \forall x, y \in X$ (p-norm)

$\Rightarrow d_{\|\cdot\|, p}(x,y) = \|x-y\|^p$ is an invariant metric on X inducing the same topology as the orig. quasi-norm

$M_X = \{p \in (0,1] : \exists$ equiv p-norm on $X\}$

$r_X = \sup M_X \Rightarrow M_X = (0, r_X)$ or $(0, r_X]$

Example: $0 < p < 1$, $\dim L_p(\mu) = \infty$

$\Rightarrow M_{L_p(\mu)} = (0, p]$, $\|\cdot\|_p$ is is p-norm, $d_p = d_{\|\cdot\|_p, p}$

$p_X = \sup\{0 < p \leq 2 : X$ has the Rademacher type $p\}$

Theorem: X quasi Banach, $\dim X = \infty$, $X \hookrightarrow_c H$

$\Rightarrow \forall r \in M_X$ $\forall$ equiv r-norm $\|\cdot\|$ on X we have

$$s(X, d_{\|\cdot\|, r}) = \alpha(X, d_{\|\cdot\|, r}) = \min\left\{\frac{p_X}{2r}, 1\right\}$$

In particular, if X is Banach, $\dim X = \infty$, $X \hookrightarrow_c H$

$$\Rightarrow s(X) = \alpha(X) = \frac{p_X}{2}$$

Proof: $S(X, d_{\|\cdot\|}, r) \geq \min\{\frac{p_X}{2}, 1\}$

$r \in \Gamma_X$, $\|\cdot\|$ equiv r-norm on X

$p \in (0, p_X]$ s.t. X has type p

$X \underset{c}{\hookrightarrow} H \overset{\text{Randr.}}{\Longrightarrow} X \hookrightarrow L_0(\mu)$ for some μ

$\overset{\text{Nikišin}}{\Longrightarrow} X \hookrightarrow L_q(\mu)$ for any $q \in (0, p)$

$q \in (0, 2] \Longrightarrow \|\cdot\|_q^q$ is negative definite $\overset{\text{Schönberg}}{\Longrightarrow}$

$\Longrightarrow \exists$ Hilbert space H, $\exists S: L_q(\mu) \longrightarrow H$

$\|x-y\|_q^q = \|S(x) - S(y)\|_H^2 \quad \forall x, y \in L_q(\mu)$

$S(X, d_{\|\cdot\|, r}) \leq \alpha(X, d_{\|\cdot\|, r})$ trivial

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$\alpha(X, d_{\|\cdot\|, r}) \leq \min\{\frac{p_X}{2r}, 1\}$:

Modification of a method of Austin to get an upper estimate for $\alpha(\Gamma)$ for some groups Γ

Baudier used it for l_p spaces

Recall: M, N metric spaces, $T: M \longrightarrow N$ injective

distortion $(T) = \mathrm{Lip}(T)\,\mathrm{Lip}(T^{-1})$

$$C_N(M) = \inf_{\substack{T: M \to N \\ \text{injective}}} \text{distortion}(T)$$

Lemma: Let X quasi Banach space, $r \in M_X$, $\|\cdot\|$ equiv r-norm on X, Y Banach

Suppose $(M_n, \delta_n)_{n\in\mathbb{N}}$ are finite d-discrete ($d>0$) metric spaces

s.t. • $\operatorname{diam}(M_n) \longrightarrow \infty$

• $\exists\, \gamma \in (0,1]\ \exists\, A,B>0$ s.t. $\forall n\in\mathbb{N}\ \exists\, f_n : M_n \to X$ s.t.

$$A\,\delta_n(x,y)^\gamma \le \|f_n(x)-f_n(y)\|^r \le B\,\delta_n(x,y) \quad \forall x,y\in M_n$$

• $\exists\, \eta \in (0,1]\ \exists\, K>0$ s.t. $c_Y(M_n) \ge K \operatorname{diam}(M_n)^\eta \quad \forall n\in\mathbb{N}$

$$\Longrightarrow \alpha_Y(X, d_{\|\cdot\|,r}) \le \frac{1-\eta}{\gamma r}$$

Proof: not hard

$(X, d_{\|\cdot\|,r})$ $\quad$ Y bad distortion

f_n nice embeddings M_n

Application: X quasi-Banach, $\dim X = \infty$, $r \in M_X$, $\|\cdot\|$ equiv r-norm on X

$n\in\mathbb{N}$, $M_n = \{0,1\}^n$ (Hamming cube) with the ℓ_1 metric d_1

(M_n, d_1) finite, 1-discrete, $\operatorname{diam}(M_n, d_1) = n$

Let $n\in\mathbb{N}$

Kalton, unpublished: ℓ_{p_X} is finitely representable in $(X, \|\cdot\|)$

$\Rightarrow \exists S_n : \ell_{p_X}^n \longrightarrow X$ linear s.t.

$$\|x\|_{p_X} \leq \|S_n(x)\| \leq 2\|x\|_{p_X} \quad \forall x \in \ell_{p_X}^n$$

Define $\varphi_n : H_n \longrightarrow \ell_{p_X}^n$, $(x_i) \longmapsto (x_i)$ and $f_n = S_n \circ \varphi_n : H \longrightarrow X$

$$\rightsquigarrow d_1(x,y)^{r/p_X} \leq \|f_n(x) - f_n(y)\|^r \leq 2^r d_1(x,y) \quad \forall x,y \in H_n$$

$\overset{\text{Enflo}}{\Longrightarrow} c_H(H_n, d_1) = \sqrt{n} = \operatorname{diam}(H_n, d_1)^{\frac{1}{2}}$

$$\overset{\text{Lemma}}{\Longrightarrow} \alpha_H(X, d_{\|\cdot\|}, r) \leq \frac{1 - \frac{1}{2}}{\frac{r}{p_X}} = \frac{p_X}{2r}$$

$$\Longrightarrow \alpha(X, d_{\|\cdot\|}, r) \leq \min\left\{\frac{p_X}{2r}, 1\right\} \quad \square$$

generalization for Y with Enflo type