A rough-paths type approach to non-commutative stochastic integration

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Joint work with **Aurélien DEYA** (Institut Elie Cartan)
Journal of Functional Analysis, Vol. 265, Issue 4, 594-628, 2013.

Objective : Adapt ideas from **rough paths theory** in a **non-commutative probability** setting.

$\longrightarrow$ To show the flexibility of the rough-paths machinery.

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Outline

1. Basics on classical rough paths theory
   - $\gamma > 1/2$ ('Young case')
   - $\frac{1}{3} < \gamma \leq \frac{1}{2}$
2. Non-commutative probability theory and rough paths
   - Non-commutative processes
   - Integration
   - The free Bm case

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Classical rough paths theory [Lyons (98)]

Consider a **non-differentiable** path

$$x : [0, T] \to \mathbb{R}^n,$$

with **Hölder regularity** $\gamma \in (0, 1)$ (i.e., $\|x_t - x_s\| \leq c\,|t - s|^{\gamma}$).

Question : Given a smooth $f : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, how can we define

$$\int f(x_t)\, dx_t \quad ?$$

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We would like this definition to be sufficiently extensible to cover **differential equations**

$$dy_t = f(y_t)\, dx_t.$$

Rough paths theory $\Longrightarrow$ Give a sense to

$$\int f(x_t)\,dx_t$$

when $x : [0, T] \to \mathbb{R}^n$ is a γ-Hölder path with $\gamma \in (0, 1)$.

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Application : Pathwise approach to stochastic calculus.

- The construction of the integral **depends on** γ :

$$\gamma > \frac{1}{2} \quad , \quad \gamma \in (\frac{1}{3}, \frac{1}{2}] \quad , \quad \gamma \in (\frac{1}{4}, \frac{1}{3}] \quad , \quad \ldots$$

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- When $\gamma \leq \frac{1}{2}$, **additional assumptions** on x.

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$\gamma > 1/2$ ('Young case')

Theorem (Young (1936))

If x *is* γ*-Hölder and* y *is* κ*-Hölder with* $\gamma + \kappa > 1$*, then the Riemann sum* $\sum_{t_i \in \mathcal{P}_{[s,t]}} y_{t_i}(x_{t_{i+1}} - x_{t_i})$ *converges as the mesh of the partition* $\mathcal{P}_{[s,t]}$ *tends to* 0*. We define*

$$\int_s^t y_u \, dx_u := \lim_{|\mathcal{P}_{[s,t]}| \to 0} \sum_{t_i \in \mathcal{P}_{[s,t]}} y_{t_i}(x_{t_{i+1}} - x_{t_i}).$$

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Application : $x : [0, T] \to \mathbb{R}^n$ γ-Hölder with $\gamma > 1/2$.
f smooth $\Rightarrow (t \mapsto f(x_t))$ γ-Hölder $\Rightarrow \int f(x_t)\, dx_t$ Young integral.

$\gamma > 1/2$ ('Young case')

Another way to see Young's result :

$$\int_s^t f(x_u)\, dx_u = f(x_s)\,(x_t - x_s) + \int_s^t [f(x_u) - f(x_s)]\, dx_u .$$

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$$\int_s^t f(x_u)\,dx_u = f(x_s)\,(x_t - x_s) + \int_s^t [f(x_u) - f(x_s)]\,dx_u.$$

Main term :

$$f(x_s)\,(x_t - x_s), \quad \text{with } |f(x_s)\,(x_t - x_s)| \leq c\,|t-s|^{\gamma}.$$

Residual term :

$$\int_s^t [f(x_u) - f(x_s)]\,dx_u \quad \text{with } |\int_s^t [f(x_u) - f(x_s)]\,dx_u| \leq c\,|t-s|^{2\gamma}.$$

As $2\gamma > 1$, 'disappears in Riemann sum'.

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$$\int_s^t f(x_u)\,dx_u = \sum_{t_i \in \mathcal{P}_{[s,t]}} \int_{t_i}^{t_{i+1}} f(x_u)\,dx_u$$

$$= \sum_{t_i \in \mathcal{P}_{[s,t]}} f(x_{t_i})\,(x_{t_{i+1}} - x_{t_i}) + \sum_{t_i \in \mathcal{P}_{[s,t]}} \int_{t_i}^{t_{i+1}} [f(x_u) - f(x_{t_i})]\,dx_u.$$

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If $x \in \mathcal{C}^\gamma$ with $\gamma \leq 1/2$, we cannot guarantee the convergence of the sum

$$\sum_{t_i \in \mathcal{P}_{[s,t]}} f(x_{t_i})\,(x_{t_{i+1}} - x_{t_i}).$$

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$\Longrightarrow$ We try to 'correct' the Riemann sum :

$$\sum_{t_i \in \mathcal{P}_{[s,t]}} \left\{ f(x_{t_i})\,(x_{t_{i+1}} - x_{t_i}) + C_{t_i,t_{i+1}} \right\}.$$

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To find out a proper C, a few heuristic considerations. Suppose that we can define $\int_s^t f(x_u)\,dx_u$. Then...

$$\begin{aligned}\int_s^t f(x_u)\,dx_u \\ &= f(x_s)\,(x_t - x_s) + \int_s^t [f(x_u) - f(x_s)]\,dx_u\end{aligned}$$

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where $|y_{s,u}| = |f(x_u) - f(x_s) - \nabla f(x_s)(x_u - x_s)| \leq c\,|u - s|^{2\gamma}$.

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$|\int_s^t y_{s,u}\,dx_u| \leq c\,|t - s|^{3\gamma}$. As $3\gamma > 1$, 'disappears in Riemann sum'.

$$\int_s^t f(x_u)\, dx_u$$

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$\Rightarrow$ **Main term :**

$$f(x_s)\,(x_t - x_s) + \int_s^t \nabla f(x_s)(x_u - x_s)\, dx_u.$$

A natural definition : when $\gamma \in (\frac{1}{3}, \frac{1}{2}]$,

$$\int_s^t f(x_u)\, dx_u \text{ " := " } \lim \sum_{t_k} \left\{ f(x_{t_k})\,(x_{t_{k+1}} - x_{t_k}) + C_{t_k,t_{k+1}} \right\}$$

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$\Longrightarrow$ to define the integral when $\gamma \in (\frac{1}{3}, \frac{1}{2}]$, we need to **assume the a priori existence of the Lévy area**

$$\mathbf{x}^{\mathbf{2},(i,j)}_{s,t} := \int_s^t \int_s^u dx_v^{(i)} dx_u^{(j)}$$

above x. Then everything works...

Assume that we can define the Lévy area $\mathbf{x}^2 = \iint dx dx$. Then :

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• We can extend this definition to $\int f(y)\, dx$ for a large class of paths y, and then **solve the equation**

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• We can extend this definition to $\int f(y)\,dx$ for a large class of paths y, and then **solve the equation**

$$dy_t = f(y_t)\,dx_t.$$

• We can show that the solution y is a **continuous function of the pair** $(x, \mathbf{x}^2)$, i.e., $y = \Phi(x, \mathbf{x}^2)$ with Φ continuous w.r.t Hölder topology.

The procedure can be extended to any Hölder coefficient $\gamma \in (0, \frac{1}{3}]$ provided one can define the iterated integrals of x :

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It applies to stochastic processes in a pathwise way.

$$Y(\omega) = \Phi\big(B(\omega), \mathbf{B}^2(\omega)\big),$$

where Φ is continuous and deterministic.

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Question : In the rough-paths machinery, what happens if we replace the γ-Hölder path

$$x : [0, T] \to \mathbb{R}^n$$

with a γ-Hölder path

$$X : [0, T] \to \mathcal{A},$$

where $\mathcal{A}$ is a **non-commutative probability space** ?

Outline

Motivation : large random matrices [Voiculescu]

Consider the sequence of random matrix-valued processes

$$M_t^n := \left(B_t^{n,(i,j)}\right)_{1\leq i,j\leq n},$$

where the $\left(B^{n,(i,j)}\right)_{n\geq 1, 1\leq i,j\leq n}$ are independent standard Bm.

Denote $S_t^n \; := \; \frac{1}{\sqrt{2}}\left(M_t^n + (M_t^n)^*\right).$

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Denote $S_t^n := \frac{1}{\sqrt{2}}\left(M_t^n + (M_t^n)^*\right)$. Then, almost surely, for all $t_1, \ldots, t_k$,

$$\frac{1}{n}\mathrm{Tr}\left(S_{t_1}^n \cdots S_{t_k}^n\right) \xrightarrow{n\to\infty} \varphi\left(X_{t_1} \cdots X_{t_k}\right),$$

for a certain path $X : \mathbb{R}_+ \to \mathcal{A}$, where $(\mathcal{A}, \varphi)$ is a particular **non-commutative probability space**. This process is called the **free Brownian motion**.

Definition : A non-commutative probability space $\mathcal{A}$ is an **algebra of bounded operators** (acting on some Hilbert space) endowed with a **trace** φ, i.e., a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ such that

$$\varphi(\mathbf{1}) = 1 \quad , \quad \varphi(XY) = \varphi(YX) \quad , \quad \varphi(XX^*) \geq 0.$$

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"To retain for the sequel" : If $X, Y \in \mathcal{A}$, then XY may be different from YX.

It is the natural framework to study the **asymptotic behaviour of large random matrices** with size tending to infinity.

Outline

Question : In the rough-paths machinery, what happens if we replace the γ-Hölder path

$$x : [0, T] \to \mathbb{R}^n \quad (\|x_t - x_s\| \leq |t - s|^\gamma)$$

with a γ-Hölder path

$$X : [0, T] \to \mathcal{A} \quad (\|X_t - X_s\|_{\mathcal{A}} \leq |t - s|^\gamma),$$

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where $\mathcal{A}$ is a **non-commutative probability space** ?

For instance, how can we define

$$\int P(X_t) \cdot dX_t$$

when P is a polynomial and $\cdot$ refers to the product in $\mathcal{A}$?

$\gamma > \frac{1}{2}$: Young theorem $\Longrightarrow$

$$\int P(X_t) \cdot dX_t := \lim \sum_{t_i} P(X_{t_i}) \cdot (X_{t_{i+1}} - X_{t_i}).$$

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$\gamma \in (\frac{1}{3}, \frac{1}{2}]$: corrected Riemann sums

$$\int P(X_t) \cdot dX_t := \lim \sum_{t_i} \{P(X_{t_i}) \cdot (X_{t_{i+1}} - X_{t_i}) + C_{t_i, t_{i+1}}\}.$$

Let us find out C in this context...

As in the finite-dimensional case, one has (morally)

$$\begin{aligned}\int_s^t P(X_u)\cdot dX_u \\ &= P(X_s)\cdot(X_t - X_s) + \int_s^t \nabla P(X_s)(X_u - X_s)\cdot dX_u \\ &\qquad + \int_s^t Y_{s,u}\cdot dX_u,\end{aligned}$$

with $\|\int_s^t Y_{s,u}\cdot dX_u\| \leq c\,|t-s|^{3\gamma}$.

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$$+ \int_s^t Y_{s,u} \cdot dX_u,$$

with $\| \int_s^t Y_{s,u} \cdot dX_u \| \leq c \, |t-s|^{3\gamma}$.

Since $3\gamma > 1$, main term :

$$P(X_s) \cdot (X_t - X_s) + \int_s^t \nabla P(X_s)(X_u - X_s) \cdot dX_u.$$

A natural definition :

$$\int P(X_t)\cdot dX_t \text{ ''} := \text{''}$$

$$\lim \sum_{t_k} \Big\{ P(X_{t_k})\cdot(X_{t_{k+1}} - X_{t_k}) + \int_{t_k}^{t_{k+1}} \nabla P(X_{t_k})(X_u - X_{t_k})\cdot dX_u \Big\}.$$

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But : Remember that in finite dimension,

$$\int_s^t \nabla f(x_s)(x_u - x_s)\cdot dx_u = \partial_i f_j(x_s) \int_s^t \int_s^u dx_v^{(i)} dx_u^{(j)}.$$

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No longer possible for $\int_s^t \nabla P(X_s)(X_u - X_s)\cdot dX_u$...
$\Longrightarrow$ **What can play the role of the Lévy area ?**

For instance, when $P(x) = x^2$,

$$\int_s^t \nabla P(X_s)(X_u - X_s) \cdot dX_u$$
$$= \quad X_s \cdot \int_s^t (X_u - X_s) \cdot dX_u + \int_s^t (X_u - X_s) \cdot X_s \cdot dX_u$$

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&= X_s \cdot \int_s^t (X_u - X_s) \cdot dX_u + \int_s^t (X_u - X_s) \cdot X_s \cdot dX_u \\
&= X_s \cdot \int_s^t \int_s^u dX_v \cdot dX_u + \int_s^t \int_s^u dX_v \cdot X_s \cdot dX_u .
\end{aligned}$$

For instance, when $P(x) = x^2$,

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$\Longrightarrow$ For $s < t$, we 'define' the Lévy area $\mathbf{X}^2_{s,t}$ as the **operator on** $\mathcal{A}$

$$\mathbf{X}^2_{s,t}[Y] = \int_s^t \int_s^u dX_v \cdot Y \cdot dX_u \quad \text{for } Y \in \mathcal{A}.$$

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$\Longrightarrow$ For $s < t$, we 'define' the Lévy area $\mathbf{X}^{\mathbf{2}}_{s,t}$ as the **operator on $\mathcal{A}$**

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Then, for every polynomial P, we can indeed define the integral

$$\int P(X_t) \cdot dX_t$$

as the limit of corrected Riemann sums involving X and $\mathbf{X}^2$.

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- $$\int f(Y_t) \cdot dX_t \cdot g(Y_t)$$

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Application : X free Bm $(\|X_t - X_s\|_{\mathcal{A}} \leq c\,|t-s|^{1/2})$

Outline

1. Basics on classical rough paths theory
   - $\gamma > 1/2$ ('Young case')
   - $\frac{1}{3} < \gamma \leq \frac{1}{2}$
2. Non-commutative probability theory and rough paths
   - Non-commutative processes
   - Integration
   - The free Bm case

Let $X : \mathbb{R}_+ \to (\mathcal{A}, \varphi)$ be a **free Bm**. In brief : to define a stochastic calculus w.r.t X, it suffices to be able to give a sense to

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We can use the results of Biane-Speicher (PTRF 98') to define (1) as an 'Itô' integral, denoted by $\mathbf{X}^{\mathbf{2},\text{Itô}} \Longrightarrow$ the 'rough-paths' calculus based on $\mathbf{X}^{\mathbf{2},\text{Itô}}$ then coincides with Itô's stochastic calculus w.r.t X.

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Another way to define (1) : consider the linear interpolation X^n of X along $t_i^n = \frac{i}{n}$ $(i = 1, \ldots, n)$, and

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Then $X^n \to X$ and $\mathbf{X}^{\mathbf{2},n} \to \mathbf{X}^{\mathbf{2},\text{Strato}}$. Moreover,

$$\mathbf{X}^{\mathbf{2},\text{Strato}}_{s,t}[Y] = \mathbf{X}^{\mathbf{2},\text{Itô}}_{s,t}[Y] + \frac{1}{2}\varphi(Y)\,(t-s).$$

Open questions

- Lévy area for other non-commutative γ-Hölder processes (with $\gamma \in (\frac{1}{3}, \frac{1}{2}])$: q-Brownian motion, $q \in (-1, 1)$.

- extension to non-martingale processes : q-Gaussian processes ([Bożejko-Kümmerer-Speicher]), ...

- smaller γ.