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Moyennabilité et théorie de Ramsey.

Amenability: topological notion.
Ramsey theory: comes from combinatorics.

I) Amenability.

A G-flow on a topological group G is a continuous action of G on a compact space.
G is extremely amenable if every G-flow admits a global fixed point:
an $x \in X$ s.t. $g \cdot x = x$ for all $g \in G$.
G is amenable if every G-flow admits an invariant Borel probability measure: for every Borel set A, $\mu(g \cdot A) = \mu(A)$.

⚠ A closed subgroup of an amenable group is not necessarily amenable.

Ex of amenable groups:
* finite groups.
* compact groups: they have a Haar measure. If the group is not compact, amenability provides "the poor man's Haar measure"
* $\mathbb{Z}$ (If $\mathbb{Z} \curvearrowright X$, then $\mathbb{Z} \curvearrowright \mathcal{M}(X)$ and there is Kakutani's fixed point theorem.)
* in fact, all abelian ~~amenable~~ groups are amenable.

Amenable groups are stable by increasing unions, closure
ex: • $\overline{\bigcup U_n} = \mathcal{U}(\ell^2)$ is amenable (and even extremely so)
• $\mathrm{Sym}(\mathbb{N}) = \overline{\bigcup \mathfrak{S}_n}$ is amenable (but not ———)

Veech: No nontrivial locally compact group is extremely amenable.

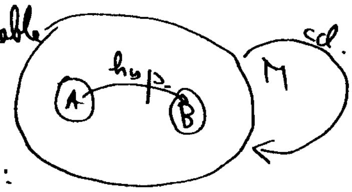
II Combinatorial description of amenability: characterisation in Ramsey theory.

Combinatorial objects: the finite subsets of automorphism groups of some structure.

Every closed subgroup of $\mathrm{Sym}(\mathbb{N})$ is isomorphic to the automorphism group of a countable homogeneous structure.
Here a structure is a set with a bunch of relations (ex: a graph with the edge relations; a ring, etc.) And the automorphisms are the bijections that

preserve these relations. A structure is homogeneous if every automorphism between finite subsets of it extends to an automorphism on the whole of it.

Every Polish group is also the automorphism group of separable metric structure. (a complete metric space; the relations R go $M^n \longrightarrow$ ~~$\{0,1\}$~~ $[0,1]$ "continuous truth values", ex: $d: M \times M \longrightarrow [0,1]$ "elements do more or less relate: $d(x,y)=0$: absolute truth $d(x,y)=1$ absolute falseness. Furthermore R is K-Lipschitz.)



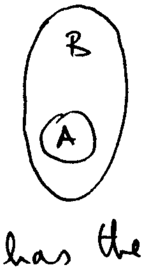
that are approximately homogeneous: If $f: A \longrightarrow B$, A, B finite, then f approximately extends: $\forall \varepsilon \; \exists g \in Aut(M) \; \forall a \in A \;\; d(f(a), g(a)) < \varepsilon$.

1) subgroups of the symmetric groups $Sym(\mathbb{N})$

Th (Moore 2011): $Aut(M)$ is amenable iff the class of all finite substructures of M, the age of M age(M), has the convex Ramsey property.

Def: A class $\mathcal{K}$ has the Ramsey property if $\forall A \in \mathcal{K} \, \forall B \in \mathcal{K} \, \exists C \in \mathcal{K}$ $\forall f$ coloring of copies of A, i.e. $f: \binom{C}{A} \longrightarrow \{0,1\}$ (isometric copies of A inside C)
$\exists \tilde{B} \in \binom{C}{B}$ $\; f|\binom{\tilde{B}}{A}$ is constant

Intuition: extreme amenability: a fixed point



a big monochromatic set.

Th (Kechris, Pestov, Todorčević, 2005)
$Aut(M)$ is extremely amenable iff age(M) has the Ramsey property.

What about amenability? It corresponds to a fixed convex combination of Dirac measures: so we are considering monochromatic convex combinations of sets. "measures are convex combinations of sets".

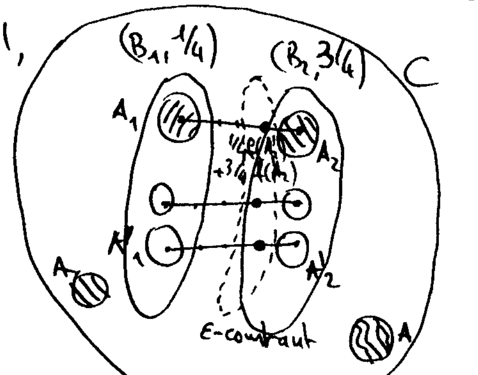
Def: A class $\mathcal{K}$ has the convex Ramsey property if $\forall \varepsilon>0\ \forall A\in\mathcal{K}\ \forall B\in\mathcal{K}\ \exists C\in\mathcal{K}$

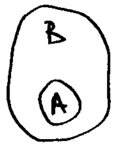
$\forall f : \mathrm{Emb}(A,C) \longrightarrow [0,1]$ $\exists \lambda_1,\dots,\lambda_n \geq 0, \lambda_1+\dots+\lambda_n=1,$

isomorphic embeddings of A inside C

$\exists \beta_1,\dots,\beta_n \in \mathrm{Emb}(B,C)$

$\forall \alpha,\alpha' \in \mathrm{Emb}(A,C)$ $\left|\sum \lambda_i [f(\beta_i\circ\alpha) - f(\beta_i\circ\alpha')]\right| < \varepsilon$





2) General Polish groups.

Let M be a metric structure and consider Aut(M)

<u>Theorem</u> (K.) Aut(M) is amenable iff age(M) has the <u>metric</u> convex Ramsey property (the previous property, where f is a <u>1-Lipschitz</u> coloring, i.e. ~~$\forall a\in A$~~ $\alpha,\alpha'\in \mathrm{Emb}(A,C)$ $|f(\alpha)-f(\alpha')| \leq \sup_{a\in A} d(\alpha(a),\alpha'(a))$

Why Lipschitz? "control of the error in the coloring": M is only approximately homogeneous: take care of epsilons: idea of Julien Melleray and Todor Tsankov. They proved that for 1-Lipschitz colorings in the context of general Polish groups.

Kechris, Pestov, Todorčević 2005: Melleray Tsankov / Moore > Kaïchouh.

Consequences: they are structural. We haven't found new amenable groups like this. But we obtain that amenability is a G_δ-notion.

<u>Theorem</u>: Let G be a Polish group; let Γ be a (discrete) countable group.

Then the set $\{\pi \in \mathrm{Hom}(\Gamma, G) \mid \pi(\Gamma)$ is an amenable subgroup of $G\}$ (wrt the induced topology)

is G_δ in $\mathrm{Hom}(\Gamma, G)$, which is a nice Polish space.

(but it is not dense, otherwise G would be amenable itself).

This is surprising, because morally amenability means: there <u>exists</u> a measure such that <u>for all</u> ... <u>there exists</u> and we are swapping quantifiers: of the form $\forall\ \exists$: much simpler.

Q: what about the space of all amenable groups?

"extreme amenability version" by M-Ts.

Ex. of nontrivial metric structure: ex: the measure algebra MALG([0,1], Leb) with $d(A,B)=\mathrm{Leb}(A\Delta B)$ whose automorphism group are the measure preserving automorphisms