On some properties of the unitary dual group.

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I. Definitions:

- A *dual group* is a unital (notion invented by Voiculescu) $*$-algebra G with $*$-morphisms $\Delta : G \to G \sqcup\!\!\sqcup G$ "coproduct", the group generated by two copies of G without relation whatsoever, $\delta : G \to \mathbb{C}$ counit, $S : G \to G$ coinverse $(S(a^*) = S(a)^*)$

s.t. 1) $(id \amalg \Delta)\circ\Delta = (\Delta \amalg id)\circ\Delta$ "coassociativity" in $G \amalg G \amalg G$

2) $(\delta \amalg id)\circ\Delta = id = (id \amalg \delta)\circ\Delta$

3) $(S \sqcup id)\circ\Delta = \delta(1_G) = (id \sqcup S)\circ\Delta$.

(here $\sqcup \neq \amalg$: if $\begin{matrix} f : A \to B \\ g : C \to D \end{matrix}$, $f \amalg g : A \amalg C \to B \amalg D$;

if $\begin{matrix} f : A \to B \\ g : C \to B \end{matrix}$, $f \sqcup g : A \amalg C \to B$)

Recall that a compact quantum group is a pair $(A, \tilde{\Delta})$ s.t. A is a C^*-algebra, $\tilde{\Delta} : A \to A \otimes A$ is a $*$-morphism st $(\tilde{\Delta} \otimes id)\circ\tilde{\Delta} = (id \otimes \tilde{\Delta})\circ\tilde{\Delta}$ and you have the quantum cancellation property: $(A \otimes 1)\Delta(A) = (1 \otimes A)\Delta(A) = A \otimes A$.

Dual groups are more noncommutative than quantum groups.

$\boxed{\text{Question}}$ Can we define a Haar state?

Definition: the *unitary dual group* is the $*$-algebra $U\langle n\rangle = *\text{-}\{u_{ij}\}_{i,j=1}^{n}$ s.t. $\sum_k u_{ki}^* u_{kj} = \delta_{ij} = \sum_k u_{ik} u_{kj}^*$ (i.e. $U = (u_{ij})$ is unitary) with $\Delta(u_{ij}) = \sum u_{ik}^{(1)} u_{kj}^{(2)}$, $\delta(u_{ij}) = \delta_{ij}$, $S(u_{ij}) = u_{ji}^*$

II Haar state.

- In the q-group case, h is a Haar state if for each $a \in A$, $(h \otimes id)\Delta(a) = h(a)1_A = (id \otimes h)\Delta(a)$.

state, i.e. a positive linear functional: $h(x^*x) \geq 0$, $h(1) = 1$.

- In the dual group case, there is no operation that combines a linear f^{al} with a morphism: we can combine $*$-morphisms and states one with another in 5 different ways: notions of noncommutative independence

for $\begin{matrix} \varphi : A \to \mathbb{C} \\ \psi : B \to \mathbb{C} \end{matrix}$ states : 1) Tensor independence : $(\varphi \otimes_T \psi)(a_{i_1} \cdots a_{i_r}) = \varphi\left(\prod_{a_{i_j} \in A} a_{i_j}\right)\psi\left(\prod_{a_{i_j} \in B} a_{i_j}\right)$

• free independence: $(\varphi \odot_F \psi)(a_{i_1} \cdots a_{i_r}) = 0$ whenever $a_{i_k} \in \mathcal{A}_{j_k}$ with $j_1 \neq j_2 \neq \cdots \neq j_r$ and the a_{i_k} are centered.

Other notions of independence: boolean, monotone, antimonotone.

The Haar state property can be stated as: $\forall \varphi$ state $(h \otimes \varphi) \circ \Delta = h = (\varphi \otimes h) \circ \Delta$

We thus state: __Definition__: h is a Haar state on $U\langle n\rangle$ if for each φ state on $U\langle n\rangle$ we have $(\varphi \odot_I h) \circ \Delta = h = (h \odot_I \varphi) \circ \Delta$

$\longrightarrow$ there are 5 notions of Haar state; __however__!

__Theorem__: • If $n = 1$, $U\langle 1\rangle$ is the algebra of polynomials on the circle $\mathbb{T}$ and the 5 notions coincide: $h(u^k) = \delta_{k0}$, $k \in \mathbb{Z}$.

• If $n > 1$, there are __no__ Haar state.

We thus __loosen__ our definition: __Def__: h is a Haar trace on $U\langle n\rangle$ if h is a trace and for each φ tracial state on $U\langle n\rangle$ we have $(\varphi \odot_I h) \circ \Delta = h = (h \odot_I \varphi) \circ \Delta$.

__Th__: • If $n \geq 2$: there are no Haar traces for boolean, monotone, antimonotone independence.

• There are free (resp. tensor) independent Haar states.

III Construction of such Haar traces

1) Free Haar trace: $(\mathcal{A}, \psi)$ a n.c.p.s with $U \in \mathcal{A}$ a Haar unitary: $\psi(U^k) = \delta_{k0}$

Recall that $(M_n(\mathbb{C}), tr_n)$ is a ncps $(\tilde{\mathcal{A}} = \mathcal{A} \sqcup M_n(\mathbb{C}), \psi \odot_F tr_n)$.

Then we compress: $\hat{\mathcal{A}} = E_{11} \tilde{\mathcal{A}} E_{11}$ (E_{11} $\begin{pmatrix} 1 & 0 \\ 0 & \end{pmatrix}$ matrix unit), $\hat\varphi = n\, tr_n \odot_F \psi$

then $j: U\langle n\rangle \longrightarrow \hat{\mathcal{A}}$ is a $*$-homomorphism
$u_{ij} \longmapsto E_{1i}\, U E_{j1} = \begin{pmatrix} u_{ij} & 0 \\ 0 & \end{pmatrix}$ (see each u_{ij} as the (i,j)-block of a Haar unitary matrix)

Then $\hat\varphi \circ j$ is a state on $U\langle n\rangle$ and it is actually the free Haar trace

Idea: $(\varphi \odot_F h) \circ \Delta(u_{i_1 j_1}^{\epsilon_1} \cdots u_{i_r j_r}^{\epsilon_r}) = (\varphi \odot_F h)(\sum_{k_1, \dots, k_r} \text{word}(k_1, \dots, k_r))$.

then use cumulants, you get $\sum_{k_1, \dots, k_r} \sum_{\substack{\mu_1 \in NC(r) \\ \mu_2 \in NC(r) \\ \mu_1 \vee \mu_2 \in NC(2r)}} K^h(\dots) K^\varphi(\dots)$ ↖ has been computed by Speicher.

If (U_t) is a unitary Lévy process, i.e.,
$U_0 = id$, $U_s^{-1} U_t \overset{(law)}{=} U_{t-s}$ for $s < t$, $U_{s_1}, U_{s_1}^{-1} U_{s_2}, U_{s_2}^{-1} U_{s_3}, \dots$ are free. Then consider

$j_t: U\langle n\rangle \longrightarrow \hat{\mathcal{A}}$: this can be seen as a Lévy process on the dual group; if
$u_{ij} \longmapsto E_{1i}\, U_t\, E_{j1}$ (U_t) is the multiplicative Brownian motion, then for $t \to \infty$, j_t goes to the Free Haar state $\longrightarrow$ similar to Riemann manifolds.

Q: Is this Haar state faithful? Haar KMS state?