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Interaction betw. top. & smooth polyhedral norms on B-spaces.

After a transfinite sequence of problems.

partially j.w. Victor Bible
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a norm depends locally on finitely many coordinates (LFC) if for each x there is a nbhd U on which $\|y\|=\|z\|$ if $y-z$ is in the kernel of f.m. given functionals.

$\|y\| = \sup\{f(y) : \|f\|=1,\ f \in \operatorname{span}(\{f_1,\dots,f_n\})\}$.

if the pool of such functionals is a given H, indep. of x, $\|\cdot\|$ is LFC-H.

Canonical example: c_0. For it we have further that the max is attained on a finite set: for $x \in S_{c_0}$, there is $\delta>0$ s.t. if m is outside that finite set, $|x(m)| \le 1-\delta$. So we can set $U = \{y : \|x-y\|_\infty \le \frac{\delta}{2}\}$ and the functionals e_n^* for $n \in F$. for $y \in U$ we have $\|y\|_\infty = \max_{n \in F} \pm e_n^*(y)$

- If Y is finite-dimensional, we can find $x \in S_Y \mapsto U_x$ and S_Y may be covered by $\bigcup U_{x_i}$.

Boundaries (subsets $B \subset B_{X^*}$ s.t. $\|x\| = \sup_{x^* \in B} x^*(x)$).

Th (Fonf-Hájek): X has a countable boundary $\Leftrightarrow$
X has a norm σ-compact boundary $\Leftrightarrow$
X has an equivalent polyhedral norm $\Leftrightarrow$
X has a LFC norm $\Leftrightarrow$
X has a LFC norm that is C^∞-smooth on $X \setminus \{0\}$.

Cor: If X has a norm σ-compact bdary, then X^* is separable and X is c_0-saturated.

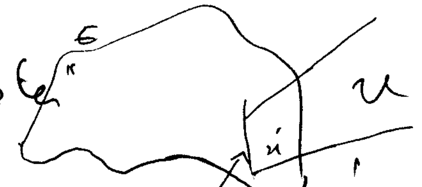
To what extent does this apply?

- Can an equivalent norm on $c_0\Gamma$ be approximated by polyhedral & C^∞-smooth norms?

$E \subset X^*$ is w^* loc rel. norm cpct: if $\forall x\, \exists U$

this is a bit like sets having "small local diameter"

(Jayne, Namioka, Rogers)



$\to$ this notion is interesting only in the nonseparable case

ex: E rel. w^*-discrete. $\forall x\, \exists U$ w^* open $U \cap E = \{x\}$, $\overline{E \cap U}^{\|\cdot\|}$ is cpct.

ex: $\{e_\gamma^*\}$, where (e_γ, e_γ^*) is a M[arkushevich] basis.

consider $E = \{f = \sum a_i e^*_{\gamma_i} : a_i \neq 0,\ \gamma_1, \dots, \gamma_n \in \Gamma\}$

take $U = \{g : |(g-f)(e_{\gamma_i})| < |a_i|\}$, so that $g(e_{\gamma_i}) \neq 0$.

then $U \cap E$ is a bdd subset of $\mathrm{span}(e^*_{\gamma_1}, \dots, e^*_{\gamma_n})$.

Idea: cover boundaries w. countable unions of such things.

If $E \subset X^*$ is both σ-w^*-LRC and σ-w^*-compact, we call it tame:

take $\bigcup \underset{w^*\text{-LRC}}{E_n^*} = E = \bigcup \underset{w^*\text{-cpct}}{K_m}$

Prop: E tame $\Rightarrow$ span (E) is tame.

$\triangle$ There is a w^*-discrete E s.t. $E+E \supseteq B_Y$ for an ∞-dim ss Y of X.

Th: If span $E \cap B_{X^*}$ is a boundary w. E tame, then X admits, up to arbitrary precision, a polyhedral LFC-norm, a C^∞-smooth LFC norm

Cor: If X has an LFC-H norm, H tame, then X has norms as above.

Examples: spaces w. σ-compact boundaries; $C(K)$ with K σ-discrete

Idea: "manage" Property $(*)$: a 1-norming set has $K = \bigcup^\infty D_n$ w. D_n rel. discrete $g(u) < 1$ if $\|u\| = 1$.

$(*)$ if for all w^*-accumulation points g of B,

$\longrightarrow$ either $\|g\| < 1$ or g doesn't attain its norm.

ex: $B = \{\pm e_n^*\} \subset c_0^*$: $\overline{B}^{w^*} = B \cup \{0\}$ and 0 certainly satisfies $(*)$

Prop: $(*) \Rightarrow \|\cdot\|$ is LFC and polyhedral.

Idea: "manage E" to get D s.t. D has $(*)$ and $|||x||| = \sup_{f \in D} |f(x)|$.

Let $B = \{f_n : n \in \mathbb{N}\}$ (distinct) be a boundary that is tame; we let $\varepsilon_n \downarrow 0$ and $D = \{(1+\varepsilon_n) f_n\}$

If $x \neq 0$, then $|||x||| > \|x\|$: there is n s.t. $\|x\| = f_n(x)$.

$|||x||| \geq (1+\varepsilon_n) f_n(x) = (1+\varepsilon_n)\|x\| > \|x\|$

$A \subset X^*$ is summable if $f \in A \Rightarrow \sum |f(e_\gamma)| < \infty$ (take $\|e_\gamma\| = 1$; $\|e_\gamma^*\| \leq K$)

Conjecture: If X has a "summable" boundary wrt an M-basis, then X has norms as above.
$\to$ true in the separable case.