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Optimal embeddings of proper metric spaces.

Embeddings: recall that for X, Y metric spaces and $f: X \to Y$,

the compression is $\rho_f(t) = \inf\limits_{d_X(x,y) \geq t} d_Y(f(x), f(y))$

the expansion is $\omega_f(t) = \sup\limits_{d_X(x,y) \leq t} d_Y(f(x), f(y))$

And f is a bilip embedding if $\omega_f(t) \leq Bt$ and $\rho_f(t) \geq At$

coarse [c] $\quad \omega_f(t) < \infty$ and $\rho_f \xrightarrow[+\infty]{} +\infty$

uniform $\quad \omega_f \xrightarrow[0]{} 0$ and $\rho_f(t) < \infty$

strong [s] $\quad$ coarse & uniform.

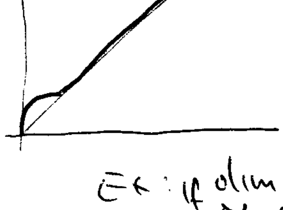
A metric space (M, d) is proper if its balls are relatively compact
stable if for bounded sequence $(x_n), (y_m)$ in M and ultrafilters $\mathcal{U}, \mathcal{V}$ we have $\lim\limits_{n \to \mathcal{U}} \lim\limits_{m \to \mathcal{V}} d_M(x_n, y_m)$
$= \lim\limits_{m \to \mathcal{V}} \lim\limits_{n \to \mathcal{U}} d_M(x_n, y_m)$

Ex: L_p is stable for all $p \in [1, +\infty]$
c_0 is not.

Baudier 2012: If M proper, X B-space, $X \supset \ell_\infty^n$'s uniformly, then $M \underset{s}{\hookrightarrow} X$.
Kalton 2007: If M is stable, then $\exists X$ reflexive with $M \underset{s}{\hookrightarrow} X$
but $\nexists X$ reflexive with $c_0 \underset{c}{\hookrightarrow} X$

Def 1 $\quad X$ is almost lip-embeddable into Y if $\exists r > 0 \; \exists D \geq 1 \; \forall \varphi: [0, \infty[\xrightarrow{\mathcal{C}} [0, 1]$ with $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$ "as close as we want"
$X \underset{aL}{\hookrightarrow} Y$ $\quad \exists f: X \to Y \quad \omega_f(t) \leq rDt \quad \rho_f(t) \geq rt\varphi(t)$.

Def 2: X is nearly isometrically embeddable into Y, $X \underset{nI}{\hookrightarrow} Y$ if $\forall \omega\, [0, \infty[\xrightarrow{\mathcal{C}}$ with $\omega(t) \geq t$ we have $\dfrac{\omega(t)}{t} \xrightarrow[0]{} \infty$
and $\forall \rho: [0, \infty[\xrightarrow{\mathcal{C}}$ with $\rho(t) \leq t$ we have $\dfrac{\rho(t)}{t} \xrightarrow[\infty]{} 0$ $\quad \exists f: X \to Y \quad \omega_f(t) \leq \omega(t)$, $\rho_f(t) \geq \rho(t)$.



Theorem. ① $p \in [1, \infty]$, X B-space.
Then X contains the ℓ_p^n's uniformly iff for all M proper subset of L^p we have $M \underset{aL}{\hookrightarrow} X$.

Ex: if $\dim X = +\infty$, M proper $\subset$ Hilbert, then $M \underset{aL}{\hookrightarrow} X$

(2) If Π is stable, then $M \underset{nI}{\hookrightarrow}$ class of reflexive B-spaces.

Proof = careful reading of Kalton.

Prop: If X is a separable B-space, then $\exists K \subset X$ compact $K \underset{aL}{\hookrightarrow} Y \Rightarrow$
Y is finitely crudely representable into X
$(\exists \lambda \geq 1 \ \forall F \subseteq Y \ \dim F < \infty \ \exists G \subseteq X \ F \overset{\lambda}{\simeq} G)$

Proof: There is a biorthogonal system in $X \times X^*$, $x_n^*(x_m) = \delta_{n,m}$, with $\overline{sp}\{x_n\} = X$.
Pick $a_n > 0$ s.t. $\sum a_n \|x_n\| \|x_n^*\| \leq 1$ and $S(x) = \sum_{n=1}^{\infty} a_n x_n^*(x) x_n$: S is compact, $\|S\| \leq 1$
$K = \overline{S(B_X)}$

(1) easy case: $f: K \underset{L}{\hookrightarrow} Y$ $\quad \|x - x'\| \leq \|f(x) - f(x')\| \leq C\|x - x'\|$.

Consider $\rightarrow$ of Heinrich-Mankiewicz '80's ~~w* cluster points of~~ points of w^*-Gateaux-d-ability
$f \circ S: B_X \longrightarrow Y \subset Y^{**}$ and $W = \{x \in B_X : f \circ S$ is w^*-Gateaux-diffable$\}$
$f \circ S$ is C-lip. There are plenty of w^*-G-d-points: $B_X \setminus W$ is Gauss-null:
there is nontrivial Gauss measure on the B-space st. this set has measure 0.
Let $x \in W$: $\|D^*_{f \circ S}(x)(k)\| \leq C\|S(k)\|$. (the norm is w^*-lsc)
But do we have $? \leq$ ________ : pick $\delta < 1$ and $k \in X$ and
$W_{\delta, k} = \{x \in W : \|D^*_{f \circ S}(x)(k)\| < \delta \|Sk\|\}$: this is also Gauss null!
Take D countable dense in X: there is $x \in W$ s.t. $\forall k \in D$ $\|D^*_{f \circ S}(x)(k)\| \geq \delta(k)$
Then this holds also for $k \in X$.
Take $h \in sp\{x_n\}$: $h = \sum_{n=1}^{N} x_n^*(h) x_n$ and $k = S^{-1}h = \sum \frac{1}{a_n} x_n^*(h) x_n$
Take $Vh = D^*_{f \circ S}(x)(k)$: $\|h\| \leq \|Vh\| \leq C\|h\|$. (here the bi $\perp$ seq is needed: to define S^{-1} on a dense set)
$X \underset{\sim}{\subseteq} Y^{**}$

(2) more difficult: assume only ~~$\exists$~~ $K \underset{aL}{\hookrightarrow} Y$. Take (R_n) increasing 2^{-n} finite net in K and $d_n = \inf_{x \neq y \in R_n} \|x - y\| > 0$: $\exists C \geq 1$ $\forall n$ $\exists f_n: K \longrightarrow Y$ $\forall x, y \in R_n$ $\|f_n(x) - f_n(y)\| \leq C\|x - y\|$.
Consider $c_n: K \longrightarrow R_n$ s.t. $\|c_n(x) - x\| \leq 2^{-n}$, and $g_n = f_n \circ c_n$.
Take $f: K \longrightarrow Y_{\mathcal{U}}$ $\quad f$ is a bilip embedding: use (1): $X \hookrightarrow Y_{\mathcal{U}}^{**}$
$x \longmapsto (g_n(x))_{\mathcal{U}}$
This implies X f.c.r. into Y: local reflexivity & properties of ultrapowers.

Construction of the embedding:

Consider $M \subset L_p$ proper and $X \supset \ell_p^n$'s.

Let $\varphi: [0,\infty[\to \mathbb{R}$ with $\varphi(0)=0$ and $\varphi(t)>0$ for $t>0$; wlog $\varphi(t)=2^{\mu(t)}$ with $\mu:[0,\infty[\to]-\infty,0]$ increasing, continuous, onto, $\lim_0 \mu = -\infty$.

There is $\sigma:]-\infty,0[\to [0,+\infty[$, $\lim_{-\infty}\sigma=0$, $\sigma(t)\le s<\sigma(t')$ iff $t\le\mu(s)<t'$.

Then $\exists f$ $\frac{1}{20}\|x-y\|_p\, 2^{\mu(\|x-y\|_p)} \le \|f(x)-f(y)\| \le 9\|x-y\|_p$ for $x,y\in M$.

for each $k\in\mathbb{Z}$ $B_k=\{x\in M: \|x\|\le 2^{k+1}\}$ is compact.

. $\exists G_{n,k}\subset L_p$ dim $G_{n,k}<\infty$ $\exists \varphi_{n,k}: L_p\to G_{n,k}$ linear $\|\varphi_{n,k}\|\le 1$

- $\forall x\in B_k$ $\|\varphi_{n,k}(x)-x\|\le \frac{\sigma(-n)}{A}$. ($L_p$ has the MAP!)

. $\exists R_{n,k}$ $G_{n,k} \underset{1+\beta}{\overset{\sim}{\hookrightarrow}} \ell_p^{d(n,k)}$ (L_p is an $\mathcal{L}^p$-space)

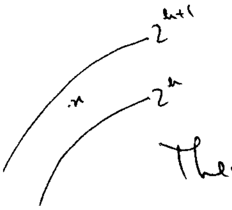
$\psi:\mathbb{Z}\times\mathbb{N}\to\mathbb{N}$ bijection. If $X\supset\ell_p^n$'s, then $\exists (H_j)$ $\ell_p^{d(\psi^{-1}(j))} \overset{S_j}{\simeq} H_j$

s.t. $(H_j)_j$ is an FDD of $H=\overline{\mathrm{sp}}\bigcup H_i$. $\to$ gliding hump argument.

Define $f_{n,k}: B_k \xrightarrow{\varphi_{n,k}} G_{n,k} \xrightarrow[\sim]{R_{n,k}} \ell_p^{d(n,k)} \overset{S_{\psi(k,n)}}{\simeq} H_{\psi(k,n)}$

$$(1-\varepsilon)\left(\|x-y\|_p - \frac{2\sigma(-n)}{A}\right) \le \|f_{n,k}(x)-f_{n,k}(y)\| \le \|x-y\|_p .$$

$$f_k: B_k \longrightarrow \sum H_{\psi(k,n)}$$
$$x \longmapsto \sum_{n=1}^{\infty} 2^{-n} f_{n,k}(x).$$



Then, for a given x, $2^k\le\|x\|\le 2^{k+1}$, take $\lambda_x = \frac{2^{k+1}-\|x\|_p}{2^k}$ and $f(x)=\lambda_x f_k(x)+(1-\lambda_x)f_{k+1}(x)$

Idea: In order to bound $\|f(x)-f(y)\|$ from below, $2^k\le\|x\|_p\le 2^{k+1}$ use $k, k+1$; $2^l\le\|y\|_p\le 2^{l+1}$ use $l, l+1$

We get: $-n-1\le\mu(\|x-y\|_p)\le -n$ $\quad$ $\sigma(-n-1)\le\|x-y\|_p<\sigma(-n)$ and so.

$2^{\mu(\|x-y\|_p)}\sim 2^{-n}$

Conjecture: If Y, X are B-spaces and Y f.c.r. into X and $M\subset Y$ proper. Does it imply that $M \underset{aL}{\hookrightarrow} X$? We need to find another gliding hump argument: in general, if a ~~sub~~space embeds, it does not necessarily in all finite codim. B-space.

Q: Do there exist X, Y B spaces with $X \underset{aL}{\hookrightarrow} Y$, $X \underset{L}{\not\hookrightarrow} Y$? Ex: Ribe: $\ell_1\oplus\left(\bigoplus \ell_{p_n}\right)_{\ell_2}$, $p_n \downarrow 1$.

This is $\simeq_{UH} \left(\bigoplus\ell_{p_n}\right)_{\ell_2}$; we have $\ell_1 \underset{s}{\hookrightarrow} \left(\bigoplus\ell_{p_n}\right)_{\ell_2}$ $\quad$ $\ell_1 \underset{aL}{\hookrightarrow} \left(\bigoplus\ell_{p_n}\right)_{\ell_2}$.