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Isométries d'algèbre de Fourier

et homomorphismes presque isométriques.

Il y a intérêt à considérer des duaux d'algèbre de groupe.

(1) Considérons G, G_1, G_2 groupe l.c.

Wendel (1951) : Si $T : L_1(G_1) \longrightarrow L_1(G_2)$ est un homom bijectif et $\|T\| \leq 1$, alors $G_1 \simeq G_2$ et on a $T(f)(g) = f(\tau(g))\chi(g)$

avec $\tau : G_2 \longrightarrow G_1$ est un homéom isomorphique de groupe et χ un caractère de G_2

Bang Johnson (1962): Considère $M(G)$.

M. Walter (1972) Si $T : A(G_1) \longrightarrow A(G_2)$ (alg de Fourier)

ou $B(G_1) \to B(G_2)$ (alg de F-Stieltjes)

et isométrique, bijectif, homomorphisme: alors $T(f)(g) = f(g_0\,\tau(g))$

avec $g_0 \in G_1$, $\tau : G_2 \to G_1$ bij homom ou antihomom.

Si G commutative, $A(G) = L_1(\hat{G})$ et $B(G) = M(\hat{G})$.

This has been generalized to Quantum groups: Daws, Pham, Spronk.

(2) Almost isometries.

Banach (1932) Stone (1937). We do not assume $T : C(K_1) \to C(K_2)$ is a homom, only that it is a linear isometry, bijective. Then

$T(f)(x) = f(\tau(x))\, g(x)$, $\tau : K_2 \to K_1$ homeom, $g \in C(K_2)$, $|g| \equiv 1$.

i.e., $T(f) = U\, S(f)$ w. $U \in C(K_2)$ unitary, S homom.

Kadison (1951): $T : A \longrightarrow B$ isometric linear bijection between C^*-algebras.

Then $T(x) = U \cdot J(x)$ where $U \in B$ unitary:

J is a Jordan homomorphism : $J(xy) + J(yx) = J(x)J(y) + J(y)J(x)$.

If furthermore A is vN, then $A = A_1 \oplus A_2$ w. $J|_{A_1}$ *-hom, $J|_{A_2}$ *-antihom

Blecher, Paulsen (2001): A, B operator algebras w. approximate identity.
If T is a c.b. isometry, bijective then $T(u) = U\,S(u)$ w. U unitary, S homom.

(3) almost isometric case: operator algebras.

Def: Banach-Mazur-distance: ...

Let X, Y be operator spaces. There is a c.b. variant.

General q: under what linear conditions do we get algebraic csq?

Th Amir (1965) - Cambern (1967): If $d(C(K_1), C(K_2)) < 2$, then $K_1 \simeq K_2$; there is a $*$-isom between $C(K_1)$ $C(K_2)$.

Roydor, Ricard (2013): let A be a nuclear sep. C^*-algebra or a vN algebra.
let B be a C^*-algebra.
There is $\varepsilon_0 > 0$ st if $d_{cb}(A, B) < 1 + \varepsilon_0$, then $A \simeq B$ as C^*-algebras.
⚠ there are nonsep C^*-algebras A, B st $A \not\simeq B$ but $d_{cb}(A, B) = 1$

(4) Kalton-Wood (1976) If $T: L_1(G_1) \to L_1(G_2)$ is a linear bijection (and a homom?!)
and $\|T\| < \gamma \approx 1{,}246$, then $G_1 \simeq G_2$. best constant.
If G_1, G_2 are abelian: then $\|T\| < \sqrt{2}$, then the groups are isomorphic and there is an explicit description; if $\|T\| < \frac{1}{2}(1+\sqrt{3})$, $\approx 1{,}3$
then $T(f)(u) = f(\tau(u)) \cdot \varphi(u)$ w. $\tau: G_1 \to G_2$ top. isom and $\varphi \in \hat{G}_2$.

(5) (w. Jean Roydor 2015): there is ε_0 st if $T: A(G_1) \to A(G_2)$ or. $B(G_1) \to B(G_2)$ are bijective, homom
→ almost isometric generalisation of Walter's result
and $\|T\|\,\|T^{-1}\| < 1 + \varepsilon_0$, then $G_1 \simeq G_2$: $T(f)(g) = f(g_0 \cdot \tau(g))$ $g_0 \in G_1$, $\tau: G_2 \to G_1$ top. isom. or antiisom.

There is a stronger result in the completely isometric case.
If $T: A(G_1) \to A(G_2)$, $\|T\|_{c.b.} < \sqrt{\frac{3}{2}} \approx 1{,}225$, we get the same
If $T: B(G_1) \to B(G_2)$, $\|T\|_{cb} < \sqrt{\frac{5}{2}} \approx$
bound of the constant: 2 or $2\sqrt{2}$.

Ideas of the proof: consider $T : A(G_1) \to A(G_2)$ ε_0-isometries;
then $T^* : A(G_2)^* = VN(G_2) \to A(G_1)^* = VN(G_1)$ is ε_0-isometric.
these are homom if ε_0 is small enough. (Royden; Ricaux).
in the cb case (One has to unitise first)

In the non cb case: almost version of ... : almost-Jordan map:
$T^*(x) = U \cdot J(x)$ with $\| J(xy) + J(yx) - (J(x)J(y) + J(y)J(x)) \| < \varepsilon_0 \|x\| \|y\|$.

If $x = \lambda_{g_1}$, $y = \lambda_{g_2}$ with $g_1, g_2 \in G_2$, this yields
$\| J(\lambda_{g_1 g_2} + \lambda_{g_2 g_1}) - (J(\lambda_{g_1}) J(\lambda_{g_2}) + J(\lambda_{g_2}) J(\lambda_{g_1})) \|_{VN(G_1)} < \varepsilon_0$.
this implies $J(\lambda_g) \overset{\varepsilon_0}{=} \lambda_{\tau(g)}$ with $\tau(g) \in G_1$ for every $g \in G_2$.
then $\tau(g_1 g_2) \overset{\varepsilon_0}{=} \tau(g_1)\tau(g_2)$ or $\tau(g_2)\tau(g_1)$ for $g_1, g_2 \in G_2$.
(We use that G lies discretely in $VN(G)$ for $\|\cdot\|$ to conclude that either τ is a homom or a homom for all (g_1, g_2)).

→ Lipschitz counterpart for c.h. variant.
Yarosh: nonlinear variant of Amir and Cambern
Dutrieux, Kalton.