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Comment reconnaître une isométrie?

The problem: Given X a metrisable space, separable, f a homeomorphism on X ($f \in \mathrm{Homeo}(X)$), how can you determine whether f is an isometry for some compatible distance on X?

Ex: If $X = [0,1]$, this is equivalent to $f^2 = 1$.

→ If $f \nearrow$, pick the orbit of an $x \in [0,1]$. $f^n(x)$ cv $\Rightarrow$ it is constant.

→ it is the identity. Conversely $|x-y| + |f(x) - f(y)|$ is a nice compatible distance.

Now let Γ be a subgroup of $\mathrm{Homeo}\, X$ ($\Gamma \leq \mathrm{Homeo}(X)$)

Say that $\Gamma \curvearrowright X$ is isometrisable if there is a Γ-invariant metric.

In general, if f is compact, we have

(i) $\Gamma \curvearrowright X$ is isometrisable

$\Updownarrow$

(ii) $\Gamma \leq \mathrm{Homeo}(X)$ is relatively compact (top of unif convergence)

$\Updownarrow$

(iii) $\Gamma \curvearrowright$ is equicontinuous: for all open $U \supseteq \Delta_X$ there is an open $V \supseteq \Delta_X$ such that $(\gamma \times \gamma)(V) \subseteq U$ for all U.

(very well known in topological dynamics)

But if X is the Baire space, $\mathrm{Homeo}(X)$ does not have a good topology anymore and (iii) is never satisfied for any group action.

Def: X, Y topological spaces. A family of maps $\mathcal{F}$ from X to Y is <u>evenly continuous</u> if $\forall x \in X\ \forall y \in Y\ \forall V$ open $\ni y\ \exists W\ y \in W \subseteq V$ $\exists U \ni x\ \forall f \in \mathcal{F}\ f(x) \in W \Rightarrow f(U) \subseteq V$.

Th (Marjanović '69) If Γ is l.c., then $\Gamma \curvearrowright X$ is isometrisable iff Γ is evenly continuous on a family of maps from X to its Alexandroff compactification

Def (Kelley, Gen. Top. book) Let X be a top. space. A family of maps $\mathcal{F}$ on X is topologically equicontinuous if $\forall x \in X\ \forall y \in X\ \forall V \ni y\ \exists W\ y \in W \subseteq V$ $\exists U \ni x\ \forall f \in \mathcal{F}\ f(U) \cap W \neq \emptyset \Rightarrow f(U) \subseteq V$.

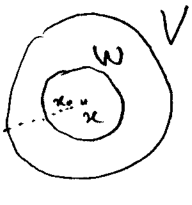
(same picture)

Consequences: suppose $\Gamma \curvearrowright X$ is top. eq.

Lemma: Assume $x_n \longrightarrow x$, $y_n \longrightarrow y$ and $\gamma_n x_n \longrightarrow y$. Then $\gamma_n^{-1} y_n \longrightarrow x$.

In particular, if $\gamma_n x \longrightarrow y$, then $\gamma_n^{-1} y \longrightarrow x$.

"Proof": We want to prove $\gamma_n^{-1} y_n \in V$. By def, we got



$x_n \in \gamma_n^{-1}(U)$ and $\gamma_n^{-1}(U) \cap W \neq \emptyset$. Then

so that $\gamma_n^{-1}(U) \subseteq V$ and in particular $\gamma_n^{-1} y_n \in V$

2nd lemma: continuity of the multiplication

We can then define an equivalence relation on X by $x \sim y \Leftrightarrow y \in \overline{\Gamma x}$
$\Leftrightarrow x \in \overline{\Gamma y}$
$\Leftrightarrow \overline{\Gamma x} = \overline{\Gamma y}$

orbits whose closures intersect are the same:

This is a closed equivalence relation.

Consider $X \longrightarrow X/\sim\ = X/\!\!/\Gamma$

Q (open): Is $X/\!\!/\Gamma$ metrisable?

Observation: If $\Gamma \curvearrowright X$ is isometrisable, $X/\!\!/\Gamma$ is metrisable.
Does this follow from top. equicontinuity?

Observation: π is continuous and open, from a 2nd countable set to a Hausdorff space.
But is $X/\!\!/\Gamma$ regular?! (Separation of a point and a closed set)

If F is closed and Γ-invariant, $x \notin F$, can we find U, V open and Γ-invariant s.t. $F \subseteq U$, $x \in V$, $U \cap V = \emptyset$?

But in the def of top. eq., we might have wanted W uniform!

Def: $\mathcal{F}$ is uniformly top. eq if $\forall y \in X\ \forall V \ni y\ \exists W\ y \in W \subseteq V\ \underline{\underline{\forall x}}$
$\exists U \ni x\ \ \forall f \in \mathcal{F}\ \ f(U) \cap W \neq \emptyset \Rightarrow f(U) \subseteq V$.

<u>Fact</u>: If $\Gamma \curvearrowright X$ is uniformly top. eq., then $X /\!/ \Gamma$ is metrisable.

Proof: take F closed Γ-invariant, $x \in X \setminus F$

Consider $V = X \setminus F$ and obtain W | and $\mathcal{O} = \bigcup_{y \in F} U_y$

$\gamma U_y \cap W_x \neq 0$
so that $\gamma y \in V = X \setminus F$
$\Gamma . \mathcal{O} \cap W = \emptyset$

Theorem: $\Gamma \curvearrowright X$ is metrisable iff it is uniformly topologically equicontinuous.

(If X is l.c., this holds iff $\Gamma \curvearrowright X$ is top. eq.)

<u>Idea of proof</u>: Assume $\Gamma \curvearrowright X$ is top. eq and there is just one $\sim$-class. (In that case, top. eq. is enough: cf Kelley's book)

Fix $x \in X$: Given U a nbhd of x define the entourage
$\mathcal{E}_U = \{(y,z) \in X : \exists \gamma\ (\gamma \cdot y, \gamma \cdot z) \in U \times U\}$ → to get a uniform structure
U → countably many if using a countable basis of nbhds.

What is its $\cap$ for all U's? the diagonal!

$\gamma_n y \to x$, $\gamma_n z \to x$ | lemma: $\gamma_n^{-1}(\gamma_n z) \to y$ which means that $y = z$.
(could reverse direction by top. eq.)

You get a dChauc.

In general: What is hidden is paracompactness; extraction of finite subcoverings:
this is an invariant version of metrisation theorems.

Variant: complete metric ... complete invariant metric
ex: $S^1 \setminus \{e^{in\alpha} : n \in \mathbb{Z}\} = X$ is a G_δ set .. there is not a compatible
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