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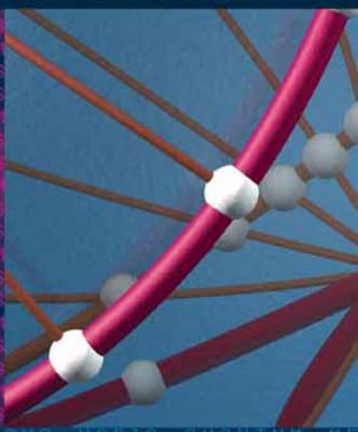

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QUO VADIS QUANTUM MECHANICS?

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Foreword

It is a great privilege for me to provide a foreword to this collection of excellent articles, written by a very distinguished assembly of authors, and put together with great skill by Nancy Kolenda, Avshalom Elitzur and Shahar Dolev. As these articles so vividly demonstrate, quantum mechanics – undoubtedly one of the supreme intellectual achievements of the 20th century – is still full of deep mysteries, despite the theory having been with us now, essentially in its modern 'final' form, for some three-quarters of a century. As we shall see, from the various articles presented here, many different viewpoints are held on how to regard the quantum formalism and its very unsettled relation to our perceived physical reality. I make no attempt to summarize these here. Instead, in this foreword, I shall address these mysteries from my own particular perspective, leaving it to the authors of these articles to present a good sampling of different viewpoints concerning the various perplexities of this subject.

As I have explained elsewhere,[1] I take the view that there are *two* quite distinct classes of quantum mystery and, although these present us with conundrums whose natures are deeply intertwined, they require, I believe, completely different attitudes towards their resolution. I call these two classes the *Z-mysteries* and the *X-mysteries*. The 'Z-mysteries' refer to *puzzle* mysteries, where it is, in my opinion, basically a matter of our accustoming ourselves to the unfamiliar nature of a genuinely strange quantum world – where individual particles seem to be able to be in two places at once or spread out over the whole of space, where the quantum spins obey beautiful algebraic and geometric laws with great precision but where these laws bear little resemblance to those of classical spinning bodies, and where the wonderful equation of Schrödinger so precisely describes quantum behaviour but in a way in which commonplace notions act according to strange mathematical procedures, momentum and energy being described by (differential) *operators*, which do not commute with the corresponding 'conjugate' classical variables (and so on). Essentially, the Z-mysteries arise from that part of the quantum formalism which is concerned with Hilbert spaces in which it is *unitary evolution* that controls behaviour. I refer to this as the **U** part of quantum mechanics.

[1] Penrose, R. (1994): *Shadows of the Mind. An approach to the missing science of consciousness* (Oxford University Press, Oxford), Chap. 5.

The remaining part of the quantum formalism has to do with the measurement paradox and the problem of the emergence of a macroscopic classical world. Here is where the 'X-mysteries' arise, these being what I call *paradox* mysteries. Indeed, rather than referring, as is more usual, to the 'measurement problem', I prefer the term 'measurement paradox', as I do not see that it can be satisfactorily resolved within the standard **U** framework of a (continuous deterministic) unitary evolution of a quantum state. Instead, we find that, in the real world of our experience, the quantum state probabilistically 'jumps' – or *reduces* – to an eigenstate of the quantum operator that is associated with the measurement being performed, according to the remaining part – the **R** part – of quantum mechanics. It is my firm belief that there can be no satisfactory resolution of the X-mysteries (including the measurement paradox itself) within the current **U** framework; that is, there can be no resolution without some fundamental *change* in the present-day quantum formalism (which holds **U** evolution to be sacrosanct). As I see it, there is no ultimate answer in the commonly held viewpoint that **R**-type behaviour arises (at least as an approximation or convenience) when a **U**-evolving measuring apparatus, with its complicated environment, becomes irretrievably entangled with the **U**-evolving quantum state under observation.[2] Accordingly, when it comes to the X-mysteries, I believe that it is not just a matter of 'accustoming ourselves to the unfamiliar nature' of quantum behaviour. We shall need a genuine *change* in the quantum theory that we use today.

Of course, although it is very easy to make a claim that some change is necessary, it is not at all easy to make plausible specific suggestions as to the nature of such a change, if we are not to run foul of observational evidence. The mass of experimental support for quantum theory is indeed highly impressive, covering an enormous range of disparate phenomena. To date, there is no substantiated experimental evidence that tells against that specific hybrid (and, strictly speaking, self-inconsistent) **U**/**R** framework that constitutes today's quantum theory. For any realistic suggested change to have a chance, it must reproduce the results of standard quantum mechanics over the whole of the broad range of circumstances where quantum mechanics has already been confirmed. Indeed, most changes that have been suggested so far (such as those of Károlyházy, Pearle, Ghirardi–Rimini–Weber), for all their ingenuity, involve a relatively small alteration to the Schrödinger equation, this normally being the addition of some sort of small stochastic term. It might be that such a change could provide a better approximation to reality than the straight Schrödinger equation, but there is relatively little clear-cut motivation for the particular forms of suggested extra term. This lack of specificity tends to apply, most particularly, to the values of the small physical constants that appear in the modified equation, these being normally

[2] Of course such a brief statement does not do justice to alternative viewpoints, such as that of 'environmental decoherence' or 'many worlds', etc.; for my own more detailed assessments, see the reference in Footnote 5, below, Chap. 29.

provided with guessed values only, chosen small enough so as not to be inconsistent with current observation, yet large enough that the modification should provide the required level of **R**-like non-unitary action.

In my own proposal, I do not supply any specific suggestion for a modified form of Schrödinger equation. Instead, I propose merely the mass/time *scale* at which changes in the standard **U** evolution must become apparent, the reasons for this proposal coming from the foundational principles of that other great revolution of 20th century physics: *Einstein's general theory of relativity*. There is a sense in which this proposed change in **U** evolution is a necessary consequence of a 'quantum gravity' that fully respects the foundational principles of general relativity, my claim being that such a 'quantum gravity' requires also a modification of quantum mechanics, not just of the classical spacetime notions of general relativity. The basic GR principle that is called upon is the *Principle of Equivalence*, (although in an earlier version of these ideas[3] the Principle of General Covariance was what was called into play). The conclusion, according to this proposal, is that a quantum superposition of two states, each state being individually completely stationary, would be *unstable*, decaying into one or the other of the constituent stationary states, giving us a physically objective **R**-type process. The average time-scale for this to take place can be calculated from a knowledge of the expectation values of the mass distributions of the two constituent stationary states, and from the specific known values of Newton's and Planck's constants.

The idea would be that any quantum measurement takes advantage of this objective **R**-type process (generalized to the more general situation in which the constituent states need not necessarily be stationary), where in many practical situations the relevant mass displacement would take place in the environment. In most situations, the resulting behaviour would be indistinguishable from that which is predicted by standard (hybrid) quantum rules, but it would be possible to measure the difference in carefully constructed experiments. Experiments aimed at eventually testing this proposal are at present under development.[4]

As a final comment, I should address the issue of human mentality and its relation to the interpretation of quantum mechanics. It is a striking fact that almost all the interpretations of quantum mechanics, that do not involve an actual change in the quantum formalism, depend to some degree on the presence of consciousness for providing the 'observer' that is required for effective realization of the **R** procedure and the consequent emergence of

[3] Penrose, R. (1996): On gravity's role in quantum state reduction, Gen. Rel. Grav. **28**, 581–600. Penrose, R. (2000): Wavefunction collapse as a real gravitational effect. In: *Mathematical Physics 2000*, ed. by A. Fokas, T.W.B. Kibble, A. Grigouriou, and B. Zegarlinski. (Imperial College Press, London), 266–282.

[4] Marshall, W., Simon, C., Penrose, R., and Bouwmeester, D. (2003): Towards quantum superpositions of a mirror. Phys. Rev. Lett. **91**, 130401 (4 pages).

a classical-like world.[5] My own position is to take issue with this, and to regard some form of an objective *physical* **R** procedure to be a necessary ingredient of an improved theory of quantum mechanics. This is not to say that I believe that the admittedly mysterious phenomenon of consciousness has no connection with the measurement paradox of quantum theory. Far from it; but my belief is that this phenomenon *depends* upon an objective form of quantum **R** procedure – not that it is *responsible* for **R**.

Wherever the truth lies, in relation to these profound issues, there are genuine prospects for important new developments in the coming century. These may come from theory, or they may arise out of the results of new experiments, or both. In whichever ways these developments come, there are exciting times ahead.

Roger Penrose June 2004

[5] See Penrose, R. (2004): *The Road to Reality: A complete guide to the Laws of the Universe* (Jonathan Cape, London), Sects. 29.7, 29.8, 34.7.

Preface

Seventeen years ago the Center for Frontier Sciences at Temple University was launched with an intriguing workshop entitled *The Philosophy of Quantum Theory*. The event, chaired by our first director, Gideon Carmi, brought together leading physicists and philosophers to address this issue. Participants included Roger Penrose, Abner Shimony, Bernard d'Espagnat, Yakir Aharanov, Basil Hiley and the late David Bohm, as the keynote speaker. In looking back, one of the workshop's greatest achievements was to establish a dialogue on quantum mechanics among these scientists, a dialogue that is still continuing today.

Quo Vadis Quantum Mechanics? is a further brainchild of the Center for Frontier Sciences. Some of the contributors to this volume were present at the first workshop, whilst others were still unknown students at that time. All contributors to the present volume were initially invited to a closed workshop/brainstorming at the Center. During that gathering, they each presented their ideas and exchanged views on what they believe are today's greatest challenges and open questions in quantum mechanics. The informal presentations were followed by lively round-table discussions. Subsequently all authors have written down, expanded and revised their contributions. To add extra spice to the book, the editors have included transcripts of some highlights of the round-table discussions. Springer did the rest – and their best.

Comparing this volume with some transcripts of the 1987 workshop at our disposal has made us very aware of the enormous progress that theoretical and experimental quantum physics has undergone during this period.

When Richard J. Fox founded the Center in 1987, he stated that leading scientists worldwide working on frontier scientific issues needed the opportunity to come together and exchange information on their work and brainstorm on their topics. The Center continues to uphold its founder's vision. Let us all keep in mind that the frontier scientific issues of the present are often the mainstream science of the future.

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In addition to the many eminent scientists whose works appear in this book, it is only fitting that we acknowledge those who assisted behind the scenes

in producing this book. We wish to thank the managing editor of *Frontier Perspectives*, Mary Huebner, the assistant editor of *Frontier Perspectives*, Meghan Nolan, the Center's coordinator, Kristine Norton, and Robert Flower and David Stein, who serve on the Center's advisory and editorial boards. We are most grateful to Angela Lahee from Springer-Verlag who guided and supported us throughout this entire production. The tireless efforts of these fine people played a vital role in producing this publication. Last but certainly not least we wish to acknowledge Richard J. Fox and his foundation for funding the workshop that provided us with the opportunity to bring all our honorable guests together and develop this fine book.

Temple University,
October 2004

Nancy Kolenda
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1 What Is the Measurement Problem Anyway? Introductory Reflections on Quantum Puzzles

A.C. Elitzur

"Can the quantum-mechanical description of physical reality be considered complete?" It is perhaps not coincidental that this question, the title of Einstein's famous onslaught on quantum mechanics [1], was echoed verbatim in the title of Bohr's reply [2]. Although Bohr opted for a "Yes", today even his ardent followers (see Wheeler below) believe that quantum mechanics is not the last word.

Someday, we all believe, a new theory will revolutionize physics, just as relativity and quantum mechanics did at the dawn of the 20th century. It will include its two parent revolutions as special cases, just as classical mechanics has been comfortably embedded within relativity theory and less comfortably within quantum mechanics. What this theory will tell us about the nature of reality is anybody's guess, but John Wheeler has vividly captured its most immediate feature [3]:

> Surely someday, we can believe, we will grasp the central idea of it all as so simple, so beautiful, so compelling that we will say to each other, "Oh, how could it have been otherwise! How could we have been so blind so long!" (p. 28)

Greenberger, however, has much more sobering reflections [4]:

> Most physicists believe that, had they been around at the birth of relativity, they would have been able to instantly appreciate its radical elements. But my own experience indicates that if Einstein were to send his paper to *Physical Review* today it would have almost no chance at all of being published. "Highly speculative!" would be the referee report, a death shell to any paper. He would have to append it to an article on string theory, or some other fad, and hope it wasn't noticed. (p. 558)

We can only hope that Wheeler is correct and Greenberger is exaggerating, and that the new theory is not already laid down in some yellowing manuscript concealed in some embittered author's drawer. Let us also hope that the theory will be published within our lifetime.

How would the puzzles of quantum mechanics fare in that revolution? Before indulging in some guesses, which are naturally bound to disclose personal biases, let us recall the puzzles themselves. There are three main questions [5]:

- **Wave–Particle Duality.** Subject any particle to an experiment set to measure waves and it will manifest unmistakably undulatory properties. Perform on it an experiment designed to measure corpuscular properties and you will end up with a particle. Both results are equivocal – and mutually exclusive. As Feynman [6] aptly remarked: the double-slit experiment (where this dual nature becomes most visible through the interference pattern) contains the core of the quantum mechanics mystery. The uncertainty principle is the general formulation of this duality, allowing only one out of a pair of physical values to be measured with arbitrary accuracy.
- **Quantum–Classical Limit.** The extraordinary predictions of quantum mechanics, such as the above interference effects, hold perfectly for particles, but fail flatly for macroscopic objects. In other words, superposition is observed in particles but never in cats, even though the latter are made of the former. Where does the jurisdiction of quantum mechanics end? Atoms also exhibit interference, and so do large molecules, although the experiments become difficult with the size of the interfering objects. Does classical mechanics simply take over at some scale [7] or is it only technological limitations that do not yet allow us to demonstrate the quantum behavior of larger objects (see Chap. 3)? This is the 'measurement problem', arising every time the properties of a particle are amplified to macroscopic extent.
- **Non-Locality.** The wavelike behavior of a single particle entails that, in order to obey conservation laws, distant parts of the wave function must instantaneously affect one another upon measurement. And indeed the violations of Bell's inequalities [8] manifest instantaneous effects of one particle's measurement on the state of another, entangled particle, regardless of the distance separating them. Quantum mechanics thus defies the spirit, if not the letter, of relativistic law.

It is such puzzles that herald a scientific revolution. Yet despite repeated promises made by superstring and other theories, no such revolution has yet appeared. Still, although we cannot know the theory itself, Wheeler's poetic sentiments about how we would *feel* upon encountering it reflect sound scientific intuition: the theory will probably *appeal* to us as true. We can therefore – and in fact, we should – lay down our expectations. It may prove to be a constructive exercise. So, based on the past experience of science, our long-anticipated theory should manifest the following qualities:

- **Beauty.** Every scientist is familiar with the aesthetic pleasure one experiences upon understanding a profound theory. An entire realm of facts becomes organically integrated, and, at the same time, simpler. Seemingly-accidental effects, which the earlier theory regarded as 'just being that way', turn out to be meaningful, even imperative. Hence, the theory that we yearn for should likewise render the quantum peculiarities just as nat-

ural as the effects known from classical physics. A consequence of this expectation of elegance is:

- **Unity.** It would, frankly, be quite disappointing if the new theory explained, say, only the wave–particle duality while non-locality was merely assumed to be there and the measurement problem was relegated to yet another revolution. Rather, one resolution should naturally entail the others.
- **Continuity.** Scientific revolutions, unlike all too many political revolutions, do not destroy the fruits of earlier theories but rather incorporate them within a new context. This is true not only for the empirical data which the earlier theories revealed, but also for many of their insights and principles, which find their place within the new framework. The new revolution will therefore incorporate not only present-day quantum formalism, but many features of its prevailing interpretations as well.[1]
- **Sacrifice.** All the above cannot come without a price. If the solution to the quantum puzzles has lingered so long, it is most likely being hindered by some highly cherished assumption which no one is willing to give up. We therefore have to prepare for a serious blow that the new theory will inflict on our world view. At this point, proponents of some of the existing interpretations might argue: "But we have already done that! We gave up the notion of objective reality and/or locality!" Well, they did, but unfortunately they did not get much in return. A genuine revolution is balanced differently: For what it has robbed us, it generously rewards us with:
- **Novel Predictions.** While the new theory will no doubt point out where we have been blind all along, as Wheeler so incisively put it, it will not stop there, but go on to tell us what is *out there* that we should now see. In other words, it will make new predictions, challenging us to verify or refute them by experiment or observation. Moreover, the theory will also yield:
- **Unexpected Dividends.** One of the most profound features of reality is that simplicity goes hand in hand with universality. One may drop a basic assumption or even an axiom and, lo and behold, the edifice built on the remaining narrower foundation turns out to be *wider*: additional phenomena, beyond those which one sought to explain, turn out to fit neatly within the new theory. Maxwell's unification of electricity and magnetism, which surprisingly turned out to account for light too, is a prominent example. Similarly, the new explanation of quantum phenomena is almost

[1] For this idea I am indebted to S. Dolev, whom I once observed analyzing a quantum-mechanical experiment in terms of a certain interpretation which I knew he was not partial to. To my inquiry he told me it has been his habit to analyze a complex quantum process in terms of several competing interpretations, as each interpretation illuminates another facet of it. See also Chap. 5 by Hartle.

> bound to illuminate some other conundrum, be it the origin of the universe [9], the nature of consciousness [10], or even something we are as yet unable to conceive of.

Having said all that, it becomes soberingly clear why none of the interpretations of quantum mechanics has won general acceptance in the physical community. To be sure, physics would be very dull had these interpretations not been proposed in the first place. They teased researchers' minds and stimulated experimentation and theorizing. Yet interpretations of quantum mechanics – especially the most ingenious ones – might sometimes do a disservice to their proponents. They might give the impression that quantum mechanics is the final word, and because they are not theories in themselves, offering no predictions that differ from quantum theory proper, they are irrefutable. This is bound to inflict barren tranquility on an over-enthusiastic adherent. Popper's [11] legacy is very instrumental in this context, and can be best appreciated when considering certain pseudo-sciences. Astrology, for example, boasts enormous explanatory power and yet, being irrefutable, is a conceptual ghost: It can never die, hence is not a living theory either. It never really *forbids* anything, hence never makes any other possibility *more likely*.

The lesson should not be lost on the quantum physicist. One should be suspicious of a framework that, instead of trying to resolve contradictions, embraces them with the aid of epistemological or methodological maneuvers, no matter how brilliantly. Contradictions have always been the lifeblood of scientific progress, and they compel us to engage upon *ontological* adventures.

Of course, "Good men must not obey laws too well" (R.W. Emerson), and neither should scientists follow too strictly any guidelines in the search for a new theory. In other words, let us remain loose enough to give Nature ample opportunity to surprise us. Einstein openly advocated a certain degree of looseness when he said that a scientist [12]:

> ... must appear to the systematic epistemologist as a type of unscrupulous opportunist: he appears as realist insofar as he seeks to describe a world independent of the acts of perception; as idealist insofar as he looks upon the concepts and theories as the free inventions of the human spirit (not logically derivable from what is empirically given); as positivist insofar as he considers his concepts and theories justified only to the extent to which they furnish a logical representation of relations among sensory experiences. He may even appear as Platonist or Pythagorean insofar as he considers the viewpoint of logical simplicity as an indispensable and effective tool of his research. (p. 684)

* * *

Participating in this volume has been a huge privilege. Perhaps the sentiments of all of us towards the subject of this volume can best be put in the words of

Rabbi Tarfon (*Ethics of the Fathers* 2, 16): "It is not upon you to complete the work, neither are you free to refrain from it."

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2 Radically Quantum: Liberation and Purification from Classical Prejudice

Hans-Peter Dürr

A hundred years after Max Planck's surprising interpretation of the puzzling features (on the basis of classical physics) in the measured spectral distribution of the black body radiation which initiated quantum mechanics, we return in this book to the question: quo vadis quantum mechanics? Should one really ask the question: where is *quantum mechanics* going? Or should we not rather ask: where are *we* going with quantum mechanics? The former skeptical question, which returns over and over again, reflects the obvious irritation of many people in view of the surprising paradoxes of quantum mechanics, which are indeed irreconcilable with our macroscopically perceived reality. But we may retort: So what? In the historical development of science, we have often experienced the situation in which former theories had to be modified because they proved to be valid only in a certain range of our experience. Regarding quantum physics, despite its uncontested empirical validity in presently known physics, the numerous verifications of newly predicted microscopic effects, and successful application in modern technologies, many still feel quite uneasy about it and urge us to 'improve' this theory or even to replace it altogether by something more familiar.

This frequently stated viewpoint takes us back to Albert Einstein's position in his famous controversy with Niels Bohr about 75 years ago. Although honored with the Nobel Prize for his early contributions to quantum mechanics, Einstein considered these paradoxes unacceptable as a satisfactory answer, in contrast to Niels Bohr and his young student Werner Heisenberg. The latter, after numerous futile attempts to remove these oddities, finally saw themselves cornered and forced to surrender, accepting the physically strange consequences of the otherwise mathematically fully consistent new mechanics. But their defeat led them to one of the greatest scientific discoveries of the modern age: quantum theory turns out to be not simply a paradigm shift in the sense of Thomas Kuhn, but does actually imply a revolutionary change in our view of 'reality', compared to the commonly accepted classical Cartesian–Newtonian reality, by rejecting its ontic character. Rather than being based on interacting objects, 'things' that 'exist', establishing a material reality (in Latin *res* means 'thing') the quantum reality (in German, 'Wirklichkeit' or actionality) is based on an immaterial and irreducible connectedness, a 'potentiality' representing a holistic, inseparable, causally un-

determined, genuinely creative world with an infinite-valued multiple logic. This, however, as can be demonstrated, constitutes no irreconcilable clash with our present world view. The commonly employed and experimentally established classical description is effectively recovered for most macroscopic systems and therefore validates the applicability of the classical reality notion in our familiar mesoscopic world.

From this viewpoint, quantum physics definitely provides a more general, more comprehensive and richer theoretical framework than classical physics, which only holds approximately under certain conditions in the limit of many quanta. We may more appropriately call the 'parts' of the 'Wirklichkeit' 'wirks' or 'haps' as minute happenings, or 'acts', as David Finkelstein [1] calls them. Hence the paradoxes of the quantum world only arise if the notions of the classical subsystem are inadequately extended to the superior system. The superior system becomes apparent particularly in microphysics (small numbers of 'wirks' or 'haps') and usually gets averaged out in meso- and macroscopic systems involving many 'wirks'. However, there may also be some surprising consequences in our much larger mesoscopic world if quantum features, by some intrinsic amplification mechanisms, manage to 'surface' and can thereby be observed and measured. In addition, it appears very likely that quantum features do somehow relate to the phenomenon of life and the spiritual dimension of our perception [2]. Because of its basic holistic structure, the quantum world offers a grand 'fulcrum', intimately connecting all fields of human experience, and in particular those which are still considered now as being completely disjoint and even mutually exclusive.

Hence I want to ask people who still have trouble accepting quantum physics the question: what kind of theory do you envisage or seek? What kind of theory would you like to advocate to replace quantum physics as a better description of physical reality? What do you mean by a 'better description' or what do you perceive as particularly 'bad' about the quantum description? What are the aspects you consider irreconcilable with macroscopic reality?

From my viewpoint there seem to be no basic problems with quantum theory as such. This does not mean that we have answers to all questions which can be asked. The problems lie more with interpretation. The question, in fact, has to be asked in a different way and directed to ourselves: how can we come to an arrangement with quantum mechanics so as to feel sufficiently comfortable or even convinced?

I would like to outline my personal perspective and approach to this question. Obviously quantum *mechanics* (QM) is still too limited. However, this is not because of its 'quantum' aspects, but rather because of its reference to 'mechanics', which only served historically as a starting point for the surprising journey into quantum physics (QPh) and still plays an important pedagogical role in this regard. It had to be and has been generalized to a relativistically invariant theory, which automatically requires the consideration of many-particle systems and quantum *field* theory (QFT). There

are still a lot of open questions but no real contradictions [see for example the Standard Model (SM) of particle physics]. The mathematical treatment of quantum field theories, however, is still unsatisfactory because, with the present mathematical tools, basic issues can only be superficially addressed. The present limitation to quasilinear systems is too restrictive and actually unjustified.

The formulation of a general, all-embracing quantum theory in terms of a relativistically invariant quantum field theory has met a number of severe difficulties in the past, which we have not been able to resolve, but only effectively tame for a certain type of theory by some formal procedures (renormalization). It seems to me that these difficulties are connected to the fact that we are still sticking too close to the classical analogues: we start from classical field theories which are subsequently 'quantized', evaluate them in a semilinear approximation (perturbation theory), and appropriately cleanse them from irritating 'divergencies' by some subtraction procedures. The whole procedure only succeeds for sufficiently 'soft' interactions. A rather baroque but up-to-now successful example is the present Standard Model. The concept of spacetime-dependent fields, the priority of propagation over interaction, the close correspondence between the constituent fields and observed 'particles' and their symmetry properties are still clear traces of our classical entry and prejudice. A future 'improved' quantum theory, must therefore, in my opinion, be more quantum rather than less, i.e., even more radical in its quantum structure. In particular it should not start with classical features, e.g., presupposing a spacetime continuum as a given general background. This 'classical' space should ultimately arise as a sufficiently valid approximation from a fundamental quantum algebra.

As a concrete example for such a more radical quantum approach, a radically unified nonlinear pure spinor theory will be sketched, the essential features of which were introduced and developed over the period 1953–1983. Viewed from the presently adopted unified quantum field theories, including the Standard Model (SM), or the supersymmetric string theories, it appears outrageously radical, but from our own point of view it should, because of its field concept, be considered only as an intermediate step, although with regard to our intuition and for our guidance, an indispensable step towards a fully-fledged quantum formulation which ultimately employs only algebraic algorithms.

2.1 The Need for a New Approach to Science Indicated by Quantum Physics

2.1.1 General Considerations

Quantum physics is significantly different in structure from classical physics. Classical physics presumes an 'ontic structure' of the world. It starts with the

question: What is? What exists? It talks about 'reality' consisting primarily of things, shaped material objects which can be distinctly localized in a 3-dimensional space and change location and form in a 1-dimensional directed time. For an explicit description we ask: what are things made of? In an attempt to separate the time-varying features from the invariant matter, we break 'things' up into ever smaller pieces, hoping to peel off their changing form from an invariant local core, the formless matter, the atoms. The form or 'gestalt' we interpret in terms of changing aggregates comprised of a large multitude of these formless atoms, i.e., localizable 'pure' particles of matter ('pure' in the sense of lacking any spatial form). The main characteristic of this approach is the priority of timeless pure particles, localized independently from each other at different space points and moving around unaware of each other.

Only in second place, because of additionally introduced interactions between these particles (this actually involves the assumption of other qualities, non-spatial 'form' features, like the electric charge of matter particles) do they become 'aware' of each other and form a common system, where 'the whole is more than the sum of its parts'. This, however, can only be described in terms of a joint system of particles if the interaction does not destroy the 'individuality' of the particles. The interaction, therefore, has to be sufficiently weak, or in mathematical terms, the equations of dynamics must be quasilinear (differential equations in time with small nonlinear, i.e., sufficiently soft interaction terms) admitting a convergent perturbation expansion. Generalizing the many-particle system to a continuum of particle densities leads to classical fields and partial differential equations describing their dynamics.

In QPh there are no 'things', but basically only connectedness. The elements are not 'pure particles' or classical fields but simply elements of connectedness which we may call 'wirks' or 'haps'. Numbers and functions of numbers are replaced by operators and operator-valued functions. 'Wirklichkeit' (or 'actionality') is no longer 'reality', but rather 'potentiality', a capability of manifesting itself as footprints in a material-energetic reality. The question is not: What is?, but rather: What is going on? What is happening? [1]. The starting point should therefore be an algebra of operators obeying the quantum commutation or anticommutation rules. Continous parameters, including time and space variables, should only show up in connection with particular representations of these algebras. The formulation of quantum field theories (QFT) in terms of dynamic equations or Lagrangians for operator-fields or operator-valued functions of such continuous (spacetime) parameters, will only occur as an effective, restrictedly valid approximation. Interaction will be more fundamental than propagation in the following sense: primarily, there is only interaction, a nonlinearity (an interacting of the field with itself), and only on a second level do we find, as a consequence of this interaction by a kind of a self-organization (constructive feedback), the effective appearance of propagators of 'particles'. These propagators will be charac-

terized by certain numerical normalization factors and also contain certain masses as softer terms. The 'particles' will be related to the empirically observed particles with the various coupling constants in their interactions with other particles being proportional to the inverse of the numerical propagation normalization factors.

In fact, the appearance of a large number of numerical constants in microphysics, in terms of mass ratios of elementary particles and coupling constants, was one of the main reasons why Werner Heisenberg in his first attempt [3] to formulate a fundamental field theory (in 1950) insisted on starting with a nonlinear theory of the non-renormalizable type. The non-renormalizable character indicates that interaction at small distances dominates propagation. As a consequence, the nonlinearity, a strong local interaction, provides for the possibility that dimensionless numbers can be generated by the dynamics. Heisenberg was familiar with this feature from his very early research work for his dissertation in 1923 on the theory of (classical) turbulent motion of fluids, where such dimensionless numbers (like those of Rayleigh and Reynold) result naturally from the nonlinear Navier–Stokes equations. As another example, the quantum mechanics of a single electron in the H atom produces infinite towers of stationary (or rather, quasi-stationary) states, and as a consequence, a corresponding infinite number of numerical energy ratios. Of course, it is exactly this close connection between relevant non-trivial results and the still continuing inability to treat such nonlinear problems successfully (as being non-renormalizable, i.e., no decoupling at small distances and hence non-applicability of perturbation theory), that has led to the broadly shared opinion that Heisenberg had definitely failed in his quite novel and highly ambitious approach.

The main purpose of the present paper is to retrace the Heisenberg approach to a fundamental quantum theory of reality in terms of the nonlinear spinor field theory of elementary particles, which he started in 1950, more than fifty years ago, and in which I actively participated for 25 years, from 1958 to 1983. Indeed, I am still involved. From my present vantage point, I consider the field theoretical formulation, as I mentioned before, as a kind of intermediary step towards an ultimate comprehensive unified theory, an in-between formulation which offers the possibility to establish approximation schemes, although perhaps only on a rather poor level (e.g., of the Tamm–Dancoff or Bethe–Salpeter type) for actually calculating mass ratios and coupling constants from basic principles. These considerations are not only of historical value but offer, to my mind, the chance to point out again the much more courageous and radical character of the approach suggested by Heisenberg et al. for the formulation of a unified theory of matter than the ones favored at present, with the Standard Model of elementary particles as the most prominent example. Clearly, the Standard Model is very successful in interpreting, or at least being without exception consistent with the extensive range of experimental data now available. Still, in the mind of most

particle physicists, the SM could hardly be said to represent anything like a 'final solution' for the fundamental problems of the material world and its dynamics.

To avoid misunderstandings at this point, let me state this more clearly. We should not and we are not aiming at an ultimate 'Weltformel', as our own approach was phrased by newspapers early in 1958. No! Quantum physics, being non-ontic, excludes such a possibility altogether. It is an 'open', largely undetermined theory. In this and many other points I fully agree with what Henry Stapp has said on this issue [4]. But the present Standard Model formulation, in our opinion, beyond this necessary openness and indetermination, has some limitations, in principle. It is more like what one might call an 'engineering model', because, to define the theory uniquely, we have to supply in an ad hoc manner numerous features concerning the character of the dominant fields and also many dimensionless numbers (related to mass ratios and coupling constants) without providing any satisfactory hint of their origin. The 'anthropic principle' – i.e., suggesting that numbers are thrown in arbitrarily by 'God' with the Big Bang at the beginning of the universe but filtered as an 'end of the pipe' condition so to speak by the requirement of coexistence of these specific numbers with the existence of the human being as observer – is hardly a hint but more of a 'poor man's' excuse.

Of course, there is nothing wrong with scientists being modest regarding their actual capability and general claims to offer explantions for any and all phenomena. But it appears unnecessarily fatalistic to use this as a starting point. Anyway, I definitely prefer to compare models, like the SM, with a van der Waals potential model in atomic physics, which allowed rather successful approximations for calculating the spectra of the heavier atoms generated by outer shell electrons by using effective classical van der Waals potentials instead of the electric potential of the atomic nucleus shielded by the electrons of the inner shells. Of course, the real breakthrough in QPh occurred much earlier with the exact calculation of the simple hydrogen atom by Wolfgang Pauli using operator algebra. The trouble repeating this success story for the more general and more complicated problems of relativistic elementary particle dynamics is that there does not seem to be any a nalogue of the simple hydrogen atom, where we could explicitly detect and convincingly isolate and extract the basic principles.

It may well be, however, that the basic principles are in fact already known: they are the principles of quantum physics. Since physics in the old meaning barely survives, we may just refer to it as the quantum principle. The trouble we meet in quantum physics and the difficulties we still have in understanding many features of it, e.g., in the context of the Standard Model, may very well be connected with our hesitation to take quantum principles really seriously. This is the reason why I have given this contribution the rather challenging title: Radically Quantum.

Before going into more detail, let me briefly indicate some features where the classical prejudice is most visible in the presently favored approach.

2.1.2 Classical Egg Shells in Quantum Physics Today

In a way, present quantum physics still looks like a kind of formal deformation of classical physics. We start from classical considerations with a classical Lagrangian or Hamiltonian or the corresponding classical equations of dynamics. We then 'quantize' the complementary canonical variables according to Heisenberg by replacing the classical variables by the corresponding operators obeying certain commutation rules consistent with the canonical theory (replacing Poisson brackets by commutators or in a Grassmann algebra by anticommutators). We limit calculations to cases which can (at least in principle) be solved explicitly. We extend these models to ones where additional interaction terms are included which can be handled as a small perturbation using the perturbation theory expansions. The corresponding space representation of the solutions leads to the Schrödinger wave equation. They are classical field equations, however, for complex fields of space and time, in contrast to the real classical fields. And there are new ones, without classical analogy, the spinor fields. These fields are again quantized (sometimes misleadingly called second quantization) where spacetime remains a classical background field, like the time parameter in quantum mechanics, on which the quantum field 'lives'.

In the case of interaction, the quantum field formulation is the only consistent way to incorporate relativistic invariance. This is connected with the possibility of pair creation and the severe consequence that the one-particle sector is intimately coupled to the many-particle sectors. Only free field dynamics expressed in term of linear operator-valued field equations can be solved exactly. More complicated field equations containing interaction terms (non-linearities) can only be solved if the interaction is sufficiently 'soft' at small distances (less dominant than the uncertainty fluctuations of the free particle at small distances) and hence can be treated as a small perturbation using the perturbation theory expansion as a valid approximation (superrenormalizable or even to a certain extent renormalizable theories).

Spacetime in these field theories is like an external classical background field. With its metric structure being intrinsically connected to the gravitational field as formulated by Einstein's general theory of relativity, it should be quantized as well to avoid inconsistencies. To quantize by starting from classical general relativity using the canonical procedure leads to a nonrenormalizable theory which does not justify perturbation expansion even for an extremely weak coupling of the gravitational field to matter fields. But there are other features which suggest a rather different approach for incorporating gravitation into quantum theory. This will be discussed later.

Altogether the procedure which starts with the phenomenological particles, i.e., the particles which appear if they are far away from each other

(asymptotic domain), and represents them by the corresponding local quantum fields, which on the contrary, characterize the behavior at very small distances, appears too luxurious (candidates to be cut off by Occam's razor) for the formulation of a fundamental theory. Such a theory should certainly also be able to generate 'particles', which are not basic entities but rather something like 'bound states' of a much smaller number of building-block fields. The presently highly favored Standard Model is a hybrid model using as basic fields, firstly, fields related to phenomenologically established particles, but also, secondly, other basic fields, like the quarks and the gluons, which do not show up themselves as particles but only play the role of constituent fields of compounds, related to other well-known particles, the strongly interacting and heavy hadrons. String theory starts solely from constituent constructs, strings, but again from the extreme classical end, with generalizations of classical theories similar to general relativity where the emphasis is on the geometric properties and we are guided by some highly attractive mathematical properties. In this context, quantum theory will only be grafted onto the classical formulation afterwards by the usual quantization procedure. From my point of view, string theory is an extremely luxurious starting point which leaves, as I see it, too much arbitrariness for selecting a specially distinguished and plausible form for the fundamental theory.

2.2 Additive Unified Quantum Field Theories

2.2.1 General Remarks

Let me start by describing briefly the presently most favored unified field theory models, which I call the additive unified quantum field theoretical models. The AUQFT models started around 1961 in the wake of the 'radically unified quantum field theoretical models', initiated essentially by Werner Heisenberg in 1950 [5]. The latter theories, RUQFT, will be treated in the next section. In fact, this short presentation with a description of their key consequences will constitute the main purpose of this contribution. Although historically they were devised later, I choose to start with the AUQFT models because they include, in particular, the so-called Standard Model (SM), which appears to be consistent with all the presently known experimental data.

The AUQFT models belong to the category of phenomenologically guided field theoretical models, in the sense that they are closely constructed from well-established subtheories connected with known particles or interactions. They differ, however, in the way this additive patchwork is glued together using general principles, mainly group theoretical considerations, to achieve some kind of unification in order to minimize the number of arbitrary fudge factors, like appropriate masses and coupling constants which are left undetermined by these theories. Like 'engineering models', they are sufficently

satisfactory for practical purposes, allowing detailed calculations and predictions for experimental outcomes – and this is indispensable for establishing sound theories – but they are barely acceptable as the final theoretical answer to the very ambitious questions we expect to be answered by a fundamental unified theory of matter, at least if we take the viewpoint of Albert Einstein, as expressed in his autobiographical notes [6]:

> Before I enter upon a critique of mechanics as the foundation of physics, something of a broadly general nature will first have to be said concerning the points of view according to which it is possible to criticize physical theories at all.
>
> The first point of view is obvious: the theory must not contradict empirical facts. However evident this demand may in the first place appear, its application turns out to be quite delicate. For it is often, perhaps always, possible to adhere to a general theoretical foundation by securing the adaption of the theory to the facts by means of artificial additional assumptions. In any case, however, this first point of view is concerned with the confirmation of the theoretical foundation by the available empirical facts.
>
> The second point of view is not concerned with the relation to the material of observation but with the premises of the theory itself, with what may briefly but vaguely be characterized as the 'naturalness' or 'logical simplicity' of the premises (of the basic concepts and of the relations between these which are taken as a basis). This point of view, an exact formulation of which meets with great difficulties, has played an important role in the selection and evaluation of theories since time immemorial. The problem here is not simply one of a kind of enumeration of the logically independent premises [...], but that of a kind of reciprocal weighing of incommensurable qualities. Furthermore, among theories of equally 'simple' foundation that one is to be taken as superior which most sharply delimits the qualities of the system in the abstract (i.e., contains the most definite claims) [...].
>
> The second point of view may briefly be characterized as concerning itself with the 'inner perfection' of the theory, whereas the first point of view refers to the 'external confirmation'. The following I reckon as also belonging to the 'inner perfection' of a theory: We prize a theory more highly if, from the logical standpoint, it is not the result of an arbitrary choice among theories which, among themselves, are of equal value and analogously constructed.

One may argue, of course, whether this point of view should be regarded as universally valid. Einstein himself only applied it 'to such theories whose object is the totality of all physical appearances'. On the basis of a more pragmatic and positivistic attitude where functionality is predominant, many scientists today actually believe that such an expectation is too idealistic in

the Platonic sense and should be given up as antiquated. They may tend to favor an anthropic principle according to which an apparent theoretical non-uniqueness of possible world models is dramatically reduced to a very small number of models which require, as a severe limitation, the compatibility of the existence of the universe with the existence of *homo sapiens sapiens* asking all these intricate questions. It is true that we should not be so arrogant as to demand rules about how this universe should be constructed and run, but I find it a rather easy and lazy attitude to discard or externalize such fundamental questions right from the start. There is no question in my mind that there will be no 'world formula' for the universe in the sense, as some people imagine, giving us very precise and unique answers to all our questions. In fact, quantum physics has taught us that the laws of nature are of a much more general form than we ever expected or imagined, leaving enough room for an infinite number of different solutions reflecting the tremendous diversity of structures, forms and processes we observe around us.

To use a more limited but quite illustrative example for this situation, let us look at the large number of light spectra of different atoms and molecules with their complicated sequences of spectral lines and continua, which serve as precise fingerprints for their existence and structure. There are long shelves of books in our libraries with tables of the measured frequencies of the emitted and absorbed light waves in these spectra. The ratio of these frequencies, corresponding to ratios of energies of states, constitute a huge collection of (dimensionless) numbers demanding to be explained by an appropriate theory for atoms and molecules. These problems can be considered to be solved, in principle, by quantum mechanics. But apart from a few very simple cases, in particular the hydrogen atom, or the H_2^+ molecule, and in a rather good approximation for some higher atoms (replacing the atomic nucleus by an effective van der Waals potential for the nucleus shielded by the electrons of the inner shells), this has not actually been carried out explicitly. Nobody is bothered by this because we have full confidence that, with the present theory, this could be demonstrated to a satisfactory degree, if necessary.

The Standard Model of elementary particles could be regarded as a van der Waals approximation to a perhaps much simpler basic theory. But we do not know this, because in particle theory we have the disadvantage that there is no simple example like the hydrogen atom to play around with. But even without the existence of such a simple system where this can be explicitly demonstrated, it may not be unreasonable to believe that there does exist such a simple underlying theoretical structure. I would guess that, even without knowledge of the H atom, scientists would not have suggested an anthropic principle to explain the huge number of spectral lines or the energy levels of the atoms, the differences of which relate to the spectral lines. For the formulation of a theory, it was in this context important not to cling to the large number of phenomenologically apparent spectral lines

themselves, but to move conceptually one level below to the electrons, where each is capable of generating an infinite number of eigenstates.

2.2.2 Phenomenologically Guided Field Theoretical Models

Elementary particle theories start from the concept of a classical particle which in a relativistic quantum theory is represented via Heisenberg's interpretation of Bohr's correspondence principle by an operator-valued quantum field $\psi(x)$ depending on a classical (c-number-type) spacetime position x. The position space is important to define a local operator interaction by a multilinear or nonlinear product of local field operators $\cdots\psi_{n+1}(x)\cdot\psi_n(x)\cdots$. The sequential operator products express a time-sequence of the operation (in our mathematical convention, the later $n+1$ is to the left of the earlier n). Hence time is actually represented twice although in a different form: firstly, in the classical fashion as the 0 th component of $x \equiv x^\mu$, and secondly, algebraically, as 'bare time' through an ordering parameter in the operator product. For a free particle, the x-label has only secondary importance and is more conveniently replaced via Fourier transformation by the 4-momentum p, with the restriction $p^2 = m^2$ relating to the rest mass of the particle. The free motion is expressed by the kinetic term in the Lagrange–Hamilton formulation or the propagator in the Feynman S-matrix expansion.

The phenomenological approach starts with phenomenologically known and, in a theoretical description, asymptotically surviving (stable) particles, and associates these with an appropriate tensor-type or spinor-type particle-operator field depending on their intrinsic spin properties (integer: tensorial, half-integer: spinorial). A practical difficulty arises over how to handle the large number of quasi-stable particles which only travel a finite rather than an asymptotically infinite distance before decaying. These decaying particles are usually treated like stable particles in the first approximation. As a consequence of the particle–wave duality, a virtual exchange of particles at smaller distances generates interactions with a Yukawa-type potential $\mathrm{e}^{-r/R}/(r/R)$, with a range R that is inversely proportional to the mass of the virtually exchanged particle. However, there is no one-to-one correlation between particle (asymptotic and on-energy-shell) and interaction (virtual and off-shell). In fact, the concept of interaction is more general than that of particles. This is evident for the massless fields. In particular the very prominent electromagnetic field has a non-particle infinite-range type interaction, the Coulomb interaction, and a similar (but 4-fold) situation holds for the gravitational field.

The phenomenologically guided field theoretical models were quite popular in the early 1960s, before the large number of rather short-lived strong interaction particles (hadrons) were discovered. All of the then-known particles were treated as quasi-stable with electromagnetic and weak interaction, to which a strong interaction was added later on, assumed to be mediated by pions as the relevant force fields. For lack of appropriate tools, all interactions,

irrespective of their actual strength, had to be treated in an unsatisfactory manner as 'sufficiently weak impacts' by perturbation theory. This is hardly a viable approach.

2.2.3 Symmetries of the Dynamics and their Effective Up- and Downgrading

The invariance of the Lagrangian and the field equations (derived from the Lagrangian according to the Hamilton principle as a functional extremum of the total action) under certain symmetry transformations are of eminent dynamical importance because they lead, according to the Noether theorems, to conservation laws for certain quantities in physical processes, the Lie generators of the dynamical symmetries. Local fields, in contrast to the asymptotic particle fields, can be viewed as appropriate local parametrizations of the symmetries of the fundamental dynamics.

There are global symmetries characterized by x-independent symmetry transformations G, usually expressed in terms of Lie groups with a certain number of real-valued Lie parameters. Some of these symmetries hold strictly (for Lagrangians which are invariant under these symmetry transformations), leading to exact conservation laws. But there are other symmetry transformations which are phenomenologically only approximately valid. There are also the more general local symmetries or gauge-type symmetries, characterized by symmetries $G(\mathrm{loc})$, with x-dependent Lie parameters. In connection with kinetic terms, they generate certain well-defined (gauge-type) interactions.

The cardinal question of particle physics relates, firstly, to the origin of the particular set of symmetries of the field dynamics (the Lagrangian or the corresponding field equations) exhibited phenomenologically, and, secondly, to why some symmetries only hold approximately, indicating perhaps some second step distortion by which originally perfect symmetries are 'broken' in their phenomenological appearance.

The conservation laws of physical processes and the mass spectrum (multiplets) of the particles exhibit a rather complicated structure which at first glance does not hint at a simple fundamental dynamics characterized by some single high symmetry group. From their additive construction, reflecting this in a straightforward combination of the various symmetries of the different interactions (strong, electromagnetic, weak and gravitational), the AUQFT models lead to a 'stutter' structure:

$$G = G_1 \otimes G_2 \otimes G_3 \otimes G_4 \ .$$

In particular the factor groups may be the same group or particular subgroups thereof. Such a situation is well-known from the physics of the heavier atoms with many electrons, which are arranged in different shells because of the Pauli principle related to the anticommutativity of the spinor field operators. The electrons all move in the same rotational $O(3)$-symmetric Coulomb

potential of the point-like nucleus (disregarding its spatial extension and its spin). The fundamental symmetry group is just $O(3)$ or the covering group $SU(2)$. [In the special Coulomb case, we even have the higher dynamical symmetry $O(4)$.] If the electrons are considered to be approximately independent of each other, then the basic symmetry group in the case of N electrons seems to be inflated to the N-fold 'stutter' symmetry

$$G = \bigotimes_{1}^{N} G_n \ .$$

Here the symmetry inflation is caused by the assumed independence of the N (electron) subsystems. This is, of course, a rather poor approximation because the electrons are all correlated by the antisymmetrization of the total wave function of the electrons enforcing the Pauli principle, and in addition are affected by their mutual electric Coulomb repulsion and their magnetic dipole interactions connected with their spin properties. In fact, because of the weak spin–spin and spin–orbit interactions, their approximately decoupled spins allow an even higher $2N$-fold 'stutter' symmetry, which by taking the various weak couplings into account step-by-step as perturbations, is then consecutively broken down to the original basic $SU(2)$ characterized by the total angular momentum J.

To demonstrate explicitly the rather sophisticated approximate symmetry structure [7], the lowest energy level structure of the 5-electron boron atom is given in Figs. 2.1 and 2.2 according to the LS and jj coupling schemes. They exhibit different in-between 'broken' higher symmetries related to subshell occupations, subspin arrangements, seniority, etc. The altogether $\binom{8}{3} = 56$ energy levels of the second (less than half-filled) atomic subshell eventually split up into 12 energy levels which only have the degeneracy of the remaining exact $SU(2)$ symmetry group. It is interesting to observe that the fine structure contains split doublets as well as singlets and split triplets.

The investigation of the spectrum of atoms hence gives another interesting hint as to how symmetries can appear to be 'broken'. In particular, there is a well-established method for approximately calculating the lowest energy states of higher atoms. In the so-called optical approximation, only a single 'outermost' electron is taken into account and considered to move in an effective classical van der Waals potential of the spherically symmetric nuclear potential screened by the averaged cloud of inner (all but the outermost) electrons. This effective screened potential will not in general be rotationally symmetric and will thus exhibit orientation properties. The spin orientation degeneracy of the energy states of the singled out (optical) electron will be removed, imitating a breakdown of the basic rotational symmetry.

In the case of a quantum field theory which involves a virtually infinite number of quanta, the above consideration relates to a certain arbitrariness in separating a certain limited subsystem from the total infinite quantum aggregation, referring to the particular system to be explicitly observed and

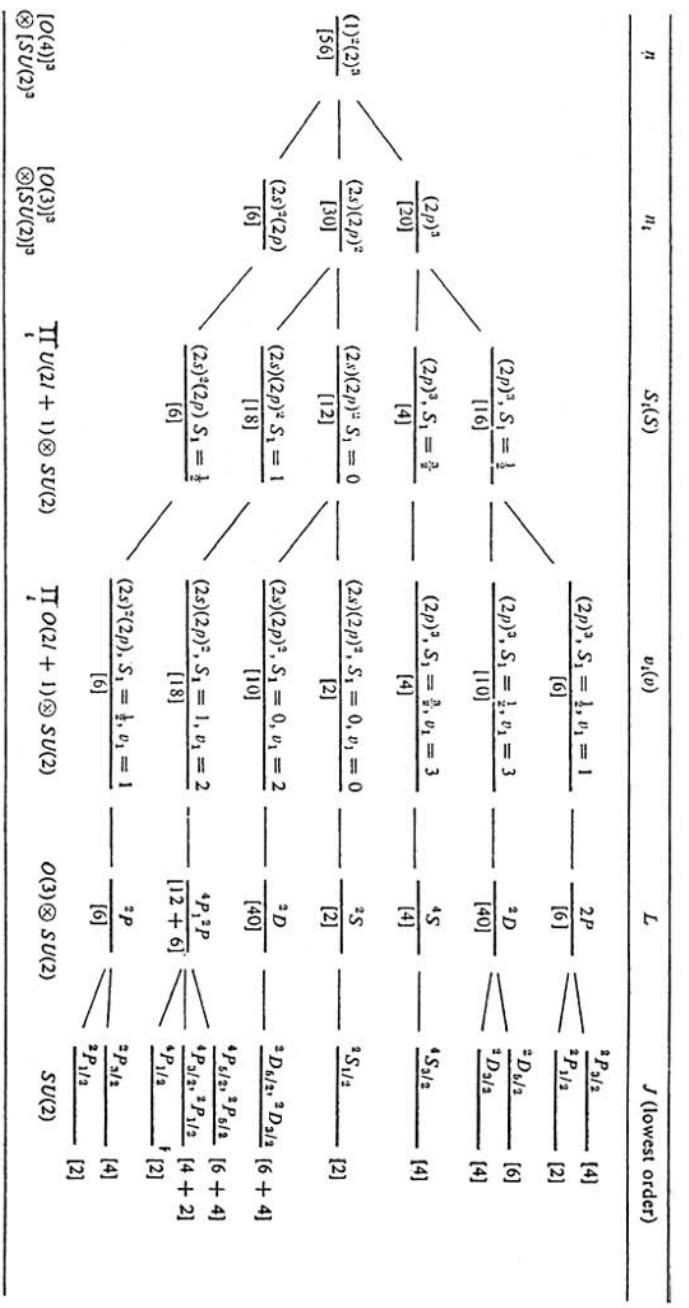

Fig. 2.1. Lowest energy level structure of the 5-electron boron atom according to the *LS* coupling scheme

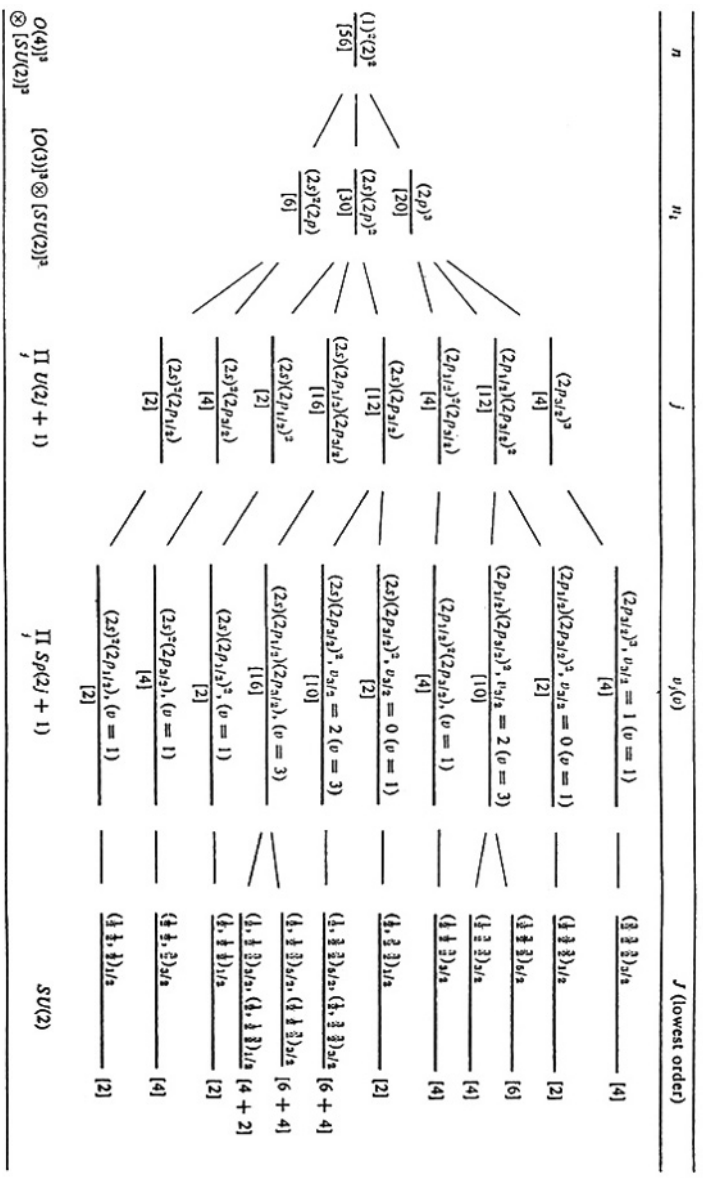

Fig. 2.2. Lowest energy level structure of the 5-electron boron atom according to the jj coupling scheme

analysed (similar to the optical electron of the atom), from the remaining, still infinite quantum system, but treated approximately as a classical 'background state' (similar to the appropriatedly shielded van der Waals potential in our atomic example). In quantum field theory this background state is usually taken as the 'vacuum' or 'ground state', characterizing the 'empty' or 'energetically lowest state', and hence 'stable state'. Mathematically, the ground state defines the representation of the quantum field algebra in the complex-valued infinite-dimensional linear state space, the Hilbert space. Different ground states define inequivalent representations of the algebra and hence disjoint Hilbert spaces. Obviously, the theoretically introduced and commonly used 'vacuum' or 'ground state' is a convenient artifact and serves as an approximation for the real physical background for our observation, which comprises the classical universe with all its galaxies, including the immediate environment of our experiment and in particular the measuring equipment and the observer, all defined in classical terms. Every measurement leading to 'facts', e.g., a blackening of a photographic film, a water droplet track (water condensation strip) of a charged particle in a Wilson cloud chamber, establishes a different inequivalent representation. The singular change of the representation by the measurement is usually interpreted as a discontinuous change, 'a collapse of the wave function' in the observed system.

Of course, simulating the 'background state' by a vacuum only works if the background is invariant under the symmetry groups of the fundamental dynamics, in particular if it is invariant under the 10-parameter spacetime Poincaré group. Due to the extreme weakness of gravitational forces and the neutralization of electric forces by opposite charges, the external influences of distant objects can generally be neglected except for some explicit nearby classical electric or magnetic force fields that are part of our measuring equipment. Such classical force fields can, however, be part of the system of observation, e.g., in the case of a ferromagnet where, due to quantum exchange forces, the electron spins can line up to produce a near-classical magnetic polarization $\boldsymbol{M}$. This state can be used as the reference state, the background state, which in this case will violate the full rotational invariance, breaking the 3-parameter rotational symmetry down to a 1-parameter rotational symmetry around the polarization axis. Different orientations of the polarization $\boldsymbol{M}$ define different Hilbert spaces which are essentially orthogonal to each other, proportional to e^{-N} with N the number of aligned spins or magnetons, and become fully orthogonal (non-equivalent) in the limit $N \to \infty$.

For infinite systems or field models, the asymmetric ground state cannot even be rotated to an infinitesimally different orientation by (angular momentum) operators defined in that Hilbert space. Only sub-infinitesimal rotations $\sim \lambda/L$ correspond to this space, relating to rotations of finite pieces of length of order λ of the ferromagnet L. They can be interpreted as wavepackets of

size λ of Bloch spin waves $\theta_{\pm}(x)$ with energy $E = \hbar c/\lambda \to 0$ with wavelength $\lambda \to \infty$, relating to massless boson modes in a relativistic formulation. As a consequence of the asymmetric background state, the whole tower of excited states will also exhibit this asymmetry.

The idea of interpreting the appearance of broken symmetries as a strictly valid fundamental symmetry but distorted by an asymmetric ground state, was discussed by Heisenberg from the very beginning of his unified theory in connection with isospin symmetry, broken by an isospin-polarized ground state in analogy with the ferromagnet or antiferromagnet. In connection with a unified theory of elementary particles, it was suggested in more general terms in the well-known but unpublished preprint of Heisenberg and Pauli [8], and in a more detailed paper by Dürr, Heisenberg, Mitter, Schlieder, and Yamazaki [9], i.e., three and four years, respectively, earlier than the papers by Goldstone [10] and Goldstone, Salam, Weinberg [11].

2.2.4 Standard Model

Within the AUQFT approach the Standard Model is essentially a phenomenologically guided unified quantum field theory with, however, basic modifications regarding the strong interactions and their corresponding particles (hadrons). These modifications were necessary to deal effectively with the fast growing number of quasi-stable hadrons (short finite half-life), which made it unreasonable to represent them all by genuine local fields. In fact, many decayed in a very short time into longer-living hadrons and eventually stable hadrons, and it appeared more appropriate to consider them as quasi-bound states or resonance states of their decay products. This suggested an approach closer to the treatment of atoms and molecules, introducing a smaller set of constituent fields. However, group theoretical considerations [12] suggested a new type of building block for hadrons: the quarks and the gluons. They were no longer simply connected to phenomenologically known particles or quasi-particles but were merely 'local interaction field constructs' representing certain basic symmetries similar to, but not identical to the constituent fields already proposed by Heisenberg in 1950 and promoted by the radically unified field theories, to be outlined in the next section. In this sense the Standard Model is a hybrid model of the phenomenologically guided field model augmented by the constituent-type-field concept.

The basic symmetry group G_{SM} of the Standard Model is a rather baroque 'stutter' symmetry, reflecting its basic 'additive' phenomenological genesis:

$$G_{\mathrm{SM}} = \underset{F'}{U(1)} \otimes \underset{Y}{U(1,\mathrm{loc})} \otimes \underset{f}{SU(2,\mathrm{loc})} \otimes \underset{c}{SU(3,\mathrm{loc})} \otimes \underset{\text{family}}{G'(?)} \otimes \underset{\text{gravity}}{G''(?)} \ ,$$

with F', Y, f, c the fermion number, hypercharge, flavor (e.g., isospin for the first family) and color properties. The majority of the groups are local groups generating gauge interactions. The 3-fold family diversity is still not satisfactorily incorporated and gravitation is left out.

Excluding gravitation and possible field additions needed to resolve the 'family' puzzle, there are altogether 142 constituent fields in the Standard Model, comprising 90 fermion fields and 52 boson fields. The fermion fields consist of three equivalent families of 15 Weyl spinor fields and their antiparticles each accounting for four 3-colored hadrons (a left-handed flavor doublet and two right-handed flavor singlets) and three colorless leptons (a left-handed flavor doublet and a right-handed flavor singlet). The generalized flavor refers to the up–down (isospin), charm–strange, and top–bottom qualites, respectively, for the three families. The $52 = 4\times(8+4)+2\times 2$ boson fields consist of an 'isospin' singlet, 8-colored, vector gauge field (gluon) and an 'isospin' doublet, uncolored, scalar Higgs field and its antiparticles. Higgs fields are required to generate the breaking of the isospin symmetry, or more generally the flavor symmetries.

In trying to unify this theory further to a Grand Unified Theory (GUT), there have been many attempts to get rid of the 'stutter' structure of the symmetry by embedding it into a higher symmetry (without taking into account family triplication, etc.). There have been various proposals [13], e.g.,

$$G_{\text{grand}} = \underset{F}{U(1)} \otimes \underset{fc}{SU(5,\text{loc})} \ .$$

The 15 spinor fields can then be grouped into a 10-plet and 5-plet (although mixed in helicity and fermion number) of $SU(5,\text{loc})$. The interaction is carried by 24 vector gauge fields, and $24+5$ Higgs fields would have to be introduced to break the grand symmetry into flavor and color subsymmetries. Obviously there will be a lot of superfluous and unaccounted-for fields.

It should be mentioned that the Standard Model and the speculative extended models require a particular mechanism to contain all 'colored' fields in such a way that they never show up asymptotically. This may be a consequence of the necessarily nonlinear interaction of non-Abelian gauge fields preventing color charges from ever being pulled apart. But the mathematical treatment is still quite unsatisfactory because it goes beyond a perturbation theory approach. Also to break symmetries in a mathematically transparent way, Lorentz-scalar Higgs fields have to be added in an ad hoc manner to enforce the existence of the desired asymmetric ground state, not to mention the large numbers of mass terms which cannot be deduced from the theory.

This may suffice to indicate that the present status of a 'comprehensive quantum theory of matter' is far from being satisfactory. The open questions may not have a definite and rigorous answer, e.g., as in the case of the anthropic principle. But it is not illegitimate to criticize the particular approach, in particular to ask: why should reality be such as to allow reductionist solutions, which depend on approximate linearizations? Because, after all, quantum theory strongly emphasizes the fundamental dominance of connectedness over separateness. It appears that we should turn the question around. Rather than ask why existing 'separated' things come together to

form various complicated compounds, we should make the opposite inquiry: why does it happen that the intrinsic tight connectedness has the tendency to favor processes of differentiation and 'emancipation' – arrangements loosening connectedness and weakening interactions by appropriate mutual compensation of forces in certain 'regions' (such as happens as a consequence of repulsive forces between opposite charges in electrodynamics), or by the 'repulsive' exchange forces enforcing the Pauli exclusion of states, or quite differently, by destructive interference – to create quasi-empty 'in-betweens', and simulate a world consisting approximately of (sufficiently weakly interacting) separate as-if-parts?

2.3 Radically Unified Field Theoretical Models

2.3.1 Historical Remarks

The occurrence of very different types and strengths of interactions is a very interesting feature of matter dynamics and there were early efforts in science to find some theoretical reasons for their particular existence and possibilities for a common origin. Being very familar in our daily life with the phenomenon of 'weight', an attraction of all matter by the earth, it was an impressive 'unification of interactions' by Isaac Newton to discover that the forces of celestial motion – described by Johannes Kepler for the planetary system as planets being attracted by the central sun – were caused by the same gravitational force which made the apple drop to the ground. The electric and magnetic interactions were combined to an electromagnetic interaction, also responsible for the chemical forces which made solid matter appear as spatially extended objects (*res extensa* of Descartes) and included later, as shown by Faraday and Maxwell, the phenomenon of electromagnetic waves. The Maxwell equations allowed an extremely compact 'unified' theoretical description in terms of an electromagnetic field generated by and interacting with electric charges and currents. The field concept could be similarly extended to gravitational interaction. Albert Einstein in his general theory of relativity [14] identified the gravitational field with metric properties of the spacetime continuum (curved pseudo-Euclidean geometry). Einstein [15] and Hermann Weyl [16] attempted later to generate a comprehensive 'unified theory' by appropriately incorporating electrodynamics into a generalized geometric structure, without much success.

Quantum physics, however, put an end to this very promising approach to a classical unified theory. Firstly, gravitation, the heart of the geometric approach, proved rather recalcitrant as regards quantization, in contrast to the electromagnetic field, but, secondly, two new interactions were discovered, the weak interaction connected with radioactive decay, and more importantly, the strong interactions relevant for atomic nuclei. With the discovery of the strongly interacting π-meson, it appeared that strong interactions could be

handled in a rather similar way to the electromagnetic one, with light quanta being replaced by pions. This favored the additive approach to a quantum field theory, as described in the last section. But due to stubborn inconsistencies at small distances (divergency problems) in the marriage of quantization and local interactions enforced by special relativity [17], the quantum field theoretical approach caused great frustration and called for a basically novel approach.

As a consequence, the majority of quantum physicists turned away from local quantum field theories altogether and looked for an appropriate description in terms of a scattering-matrix theory. This starts from the experimentally accessible asymptotically large distance region, where, in the case of finite-range interactions, only free particles occur. The S-matrix theory was developed by Heisenberg [18] in the mid-1940s as a kind of phenomenological model for an elementary particle theory starting solely from 'observable quantities'. This approach proved very powerful by the observation that the necessary condition of relativistic causality, represented in field theories by the causal condition of 'locality', could be incorporated there in terms of certain analyticity requirements for the scattering matrix.

Concerning the analyticity properties of the S-matrix theory, although they are mathematically quite attractive and in many ways productive as a substitute for the physically relevant local causality, Heisenberg felt that this new tool was intuitively not very illuminating with regard to the dismal behaviour at short distances. To avoid the obstinate divergence difficulties at small distances, he therefore returned in the early 1950s to the quantum field approach with the idea of postulating a fundamental length [3] indicating a spacetime range within which quantum theory in the presently used form would not be valid but had to be modified. This had two important consequences. Firstly, the local constituent fields would not be identical to the canonical fields appearing asymptotically on the periphery connected with the observable particles. Secondly, the coupling constant characterizing the interaction, a nonlinear term, would be proportional to a positive power of the fundamental length, hence rendering the theory non-renormalizable in the common terminology, i.e., not reducing to a free and hence canonical field theory in the limit of short distances. However, the latter troublesome consequence provided an interesting opportunity to start with a much smaller number of constituent fields compared to the number of asymptotic particle-like fields representing so-called bound states of the constituent fields. This radically new approach opened the way to a 'radically unified field theory'. The price was non-renormalizability. Did this really represent an insurmountable barrier? Or was it on the contrary the key to a new type of solution?

I emphasize this point in order to demonstrate that these RUQFTs based on non-particle-like constituent fields with possibly unusual (non-canonical) properties, developed in the early 1950s by Heisenberg [19,20] were, and still are actually more revolutionary in their basic approach than the AUQFT

constructions, including the presently accepted Standard Model. In fact, the non-canonical character (unusual transformation under dilatations) required an indefinite metric in the state space (Nevanlinna space instead of a Hilbert space) [19,20] familiar in the Gupta–Bleuler formulation of quantum electrodynamics and gravitation [21,22].

We should realize that, at the time of the conception of the RUQFT, although only the hadronic charged π-mesons had been discovered [23], Heisenberg, experiencing the many-particle production in cosmic radiation showers [24, 26, 27] clearly anticipated the later development of particle physics into a physics with a very large number of strongly interacting particles. It was only 14 years later with the work of Murray Gell-Mann [12] and others, as a reaction to the ever increasing number of hadrons and after playing around with an $SU(3)$ and an 8-fold approach to resolve the problem of *flavor* multiplicity [28], that such non-particle-type constituent fields were introduced into the AUQFT in the form of non-observable 'color' fields, the quarks and gluons, changing the AUQFT into hybrid theories, as described in the last section. Only the hadrons, in contrast to the non-hadrons, are treated in the constituent field fashion. However, they still adhere to their canonical behavior at the unobservable short distances, perhaps an unnecessary luxury because this is only required for asymptotic fields to enable a unitary S-matrix for the in- and out-states.

The further development of Heisenberg's more fundamental RUQFT approach was hindered and later totally ignored because of the phenomenologically highly successful Standard Model which, however, aimed primarily at a half-way manageable theory – convergent-renormalizable and hence accessible at small distances by perturbation theory – rather than seeking a more profound understanding of the phenomenologically exhibited dynamics and its strange symmetry pattern. As was expected, at least from the RUQFT point of view, a more profound insight into the basic dynamics would prove rather difficult if one simply tried now to improve on the present Standard Model. A more radical approach seems to be unavoidable.

But how radical should it be? New investigations along the lines of RUQFT since the early 1980s seem to indicate that the necessary changes to the familiar QFT may not be as severe as to require a dramatic modification of the basic quantum principles (e.g., the introduction of a fundamental length ℓ indicating a spacetime region where the quantum principle is grossly violated). Hence my goal in the present paper is not to compete with the SM in its results, but rather to take a different path, which allows us to reach what we believe is the heart of the problem: to establish from basic (mainly group-theoretical) considerations a fundamental highly nonlinear quantum field theory with a radically simple and unique structure which, under various more complicated conditions than hitherto assumed, permits approximate linearizations and can effectively lead to a 'baroque' phenomenological field theory of the same type as the SM. In addition, this very simple and

compact fundamental quantum theory may offer a glimpse of an even more profound level: a purely algebraic description at the foundation which avoids the spacetime-dependent fields altogether at the outset, but prepares their effective appearance, relating them to operators parameterized by the orbits of the fundamental Lie group.

2.3.2 Heisenberg–Pauli Nonlinear Spinor Theory

Observations and studies of the multiple-particle high-energy showers of cosmic rays convinced Heisenberg [17,24–26] that this could not be solely the result of many consecutive few-particle showers but must be indicative of many-particle production at very small distances connected with a rather strong interaction there. Hence it was rather unlikely that a fundamental particle theory would be convergent-renormalizable or even super-renormalizable, reflecting a free theory behavior. The divergencies in non-convergent-renormalizable and non-renormalizable theories on the other hand should not be considered a basic deficiency but simply a reflection of the fact that the canonical commutation rules commonly applied for quantization are intimately connected with the classical solutions of the free particle wave equation with a $\delta(\boldsymbol{x})$-function point source. For non-renormalizable theories the commutation relations are affected by the short-range interactions and hence have to be changed accordingly [5,39].

This led Heisenberg to suggest a quantum field theory based solely on a single, massless anticommuting (4-component) Dirac spinor field $\psi(x)$ as the basic constituent field, coupled non-linearly to itself:

$$\mathrm{i}\gamma^\mu\partial_\mu\psi(x) + \ell^2\Gamma\psi(\bar\psi\Gamma\psi) = 0\ , \tag{2.1}$$

rather analogously to the anharmonic oscillator in quantum mechanics. In this case of a canonical field (mass or inverse length dimension $\dim\psi = 3/2$), the coupling constant will have the dimension of a length-square ℓ^2 (or mass $\dim = -2$) which indicates the dominance of the interaction term over the kinetic term, identifying the theory as non-renormalizable.

Pauli [29] noticed in 1957 that the neutrino equation

$$i\gamma^\mu\partial_\mu\psi = 0 \tag{2.2}$$

is not only invariant under the $U(1)$ phase transformation, but also with regard to the extended 3-parameter group, the Pauli transformations

$$\psi \to a\psi + b\gamma_5\tilde\psi\ , \qquad |a|^2 + |b|^2 = 1\ , \qquad \tilde\psi \equiv C^{-1}\bar\psi^{\mathrm{t}}\ , \tag{2.3}$$

which later on was shown by Gürsey [30] to be isomorphic to the $SU(2)$ isospin transformations. Together with the $U(1)$ Touschek transformation [31]

$$\psi \longrightarrow \mathrm{e}^{\mathrm{i}\alpha\gamma_5}\psi\ , \tag{2.4}$$

related to a fermion number F, this results in the group

$$G = U(2) = \underset{F}{U(1)} \otimes \underset{I}{SU(2)} \tag{2.5}$$

as internal invariance group of the massless fermion equation. The surprising feature, however, was that this enlarged invariance group G_ν could be maintained by adding a nonlinear pseudovector self-interaction

$$\mathrm{i}\gamma^\mu \partial_\mu \psi + \ell^2 \gamma_5 \gamma^\mu \psi (\bar{\psi} \gamma_5 \gamma^\mu \psi) = 0 \,, \tag{2.6}$$

leading to the Heisenberg–Pauli spinor equation in 1958 [8]. This result seemed very interesting, firstly because the quartic interaction term could be fixed uniquely, and secondly, because it automatically provided the isospin group without the usual ad hoc doubling of the number of components of the spinor field. Shortly afterwards this surprise was somewhat dampened by the present author [32], with the discovery that the Heisenberg–Pauli equation can be simply interpreted as arising from an equation for a 2-component Weyl spinor field χ and the usual doubling of components, written as $\boldsymbol{\chi}$, to include isospin:

$$\mathrm{i}\sigma^\mu \partial_\mu \boldsymbol{\chi} + \ell^2 \sigma^\mu \boldsymbol{\chi} (\boldsymbol{\chi}^* \sigma_\mu \boldsymbol{\chi}) = 0 \,, \tag{2.7}$$

where the uniqueness of the quartic term prevails after doubling (in the σ-notation an isospin unit matrix is suppressed). Of course, The internal invariance group (2.5) does not suffice to embrace the empirical classification, which at that time (1958) required at least two charge-type numbers and two different fermion numbers. The electric charge was not identical with the third component of the isospin I_3 but

$$Q = I_3 + Y \tag{2.8}$$

contains in addition a hypercharge Y, which the theory (2.7) does not provide. Similarly, the fermion number has to be augmented by the introduction of an additional fermion-type number Λ such that baryon and lepton number can be distinguished:

$$B = F + \Lambda \,, \qquad L = F - \Lambda \,. \tag{2.9}$$

The difference of the non-provided quantum numbers, i.e.,

$$S = Y - \Lambda = Y - \frac{1}{2}(B - L) \,, \tag{2.10}$$

corresponds empirically to the property ‘strangeness’, the only member of the other flavor families known at the time. The corresponding larger internal invariance group would then have the extended form

$$G' = \underset{Y}{U(1)} \otimes \underset{\Lambda}{U(1)} \otimes \underset{F}{U(1)} \otimes \underset{I}{SU(2)} \,, \tag{2.11}$$

which, if basically incorporated into a spinor equation, would require a quadrupling of the spinor components to 16. Instead, at that time, the invariance group suggested was

$$G'' = \underset{\Lambda}{U(1)} \otimes \underset{F}{U(1)} \otimes \underset{\text{flavor}}{SU(3)} \,, \tag{2.12}$$

involving 12 components with fermion number doubling and extending the isospin $SU(2)$ group to a flavor $SU(3)$, not to be confused with the present color $SU(3)$.

It was actually this question, how best to accommodate the additional quantum numbers – we stuck to $SU(2)$ flavor, whilst others favored a more pragmatic approach by simply adding a sufficient number of new fields – that led to the departure of the development that finally became the well-known Standard Model. The main goal of RUQFT was different. From the point of view of the radical approach, an increase in the number of basic constituent fields was not an admissible procedure because it automatically destroyed the uniqueness of the field equation, provided that no higher internal symmetries like $SU(3)$ and higher were enforced. It is noteworthy that even in the SM, the $SU(3)$ flavor group did not survive, although only by transferring the group theoretical deficiencies by postulating the existence of three 'families', the origin of which has remained a mystery.

The new mechanism for obtaining 'additional' symmetry groups was viewed in our RUQFT approach as abandoning the requirement for the 'ground state' to be simply the 'vacuum', i.e., a representation of 'empty' or a state of pure 'nothingness', which, by definition, must have the property of being unique and invariant under the full symmetry group of the field dynamics. In fact, it was argued that the ground state should be considered as an effective 'background state' approximating everything 'outside' what is actually observed. In particular, the observed 'broken' isospin invariance reflected by the non-degenerate isospin multiplets seemed to suggest an asymmetric 'iso-ferromagnetic' or 'anti-ferromagnetic' type of polarisation of the ground state, which arose, as in superconductivity, from a Bose–Einstein condensation of isospinor–spinor pairs. Such an isospin-asymmetric ground state has two consequences:

- The occurrence of zero-mass modes similar to the Bloch spin waves of the ferro- and antiferromagnet, but here in the form of isospin-flip modes, or in modern terminology, Goldstone modes connected with the asymmetry of the ground state.
- The possibility of uncommon 'dressings' of the original bare constituent fields and their 'bound states' forming the dressed particles to be identified with the observed particles that may differ not only in their mass from

their bare modes but also by properties carried by the ground state, i.e., isospin properties.

In the original RUQFT papers [9], it was suggested that isospin-Bloch waves (zero-mass charged bosons) were eventually responsible for the appearance of the massless photons assuming an anti-isoferromagnetic background structure and neutral pairs of flip-up/flip-down modes to form photons [33]. This did not prove to be very successful.

Surprisingly, the second aspect of the unusual dressings has never been taken up in the quantum field theoretical models which dominate today, including the SM. This feature allows the introduction of 'isospin-frozen' fields, i.e., isospinor fields where the non-aligned part of the isospin is shielded by an isospin-wave dressing. They are nonlinear representations of the $SU(2)$ [34–37]. These dressings were used to give meaning to the 'spurion' introduced in 1956 by Wentzel [38] to generate strange baryons (Λ and Σ) from nucleons. In this case the hypercharge is simply the remaining I_3 charge of a frozen isopinor. The linear and nonlinear representations of $SU(2)$ cannot transform into each other, and hence define different classes of particles which are independently conserved. In addition, the dressings, as in the polaron case, will not necessarily be local but may lead to a finite spatial extension of the dressed particle (a soliton involving an infinite number of Bloch-wave bosons), in contrast to the local constituent fields. This opens a completely new way of looking at hadrons and also offers an opportunity to distinguish the different flavor families.

The close connection between asymmetric ground states and the existence of zero-mass spinless modes, although in special cases well known before, was demonstrated in 1961 for the non-relativistic case by Goldstone [10] and shortly later generalized to the relativistic case by Goldstone, Salam and Weinberg [11]. The Goldstone particles can be imagined as localized infinitesimal transformations that change the oriented ground state into an off-direction infinitesimally over a finite region, which, if infinitely extended (not localized and hence zero momentum), should not change the energy because of the formal energy degeneracy of different oriented states. The Goldstone modes are thus operator-valued Lie parameters of the broken symmetry transformations.

The asymmetry of the ground state $|\Omega\rangle$ can be expressed by the asymmetry condition of the ground state expectation value

$$\langle\Omega|\tilde{\boldsymbol{\chi}}^*\boldsymbol{\tau}^i\boldsymbol{\chi}|\Omega\rangle = \text{const.} \times \delta_3^i \tag{2.13}$$

of the isovector Higgs field $\tilde{\boldsymbol{\chi}}^*\boldsymbol{\tau}\boldsymbol{\chi}$, a scalar pair of the constituent $\boldsymbol{\chi}$ fields, in fact, the only possible local Lorentz-invariant isovector field (aside from its Hermitian conjugate) constructible from the $\boldsymbol{\chi}$. This allows the representation

$$\tilde{\boldsymbol{\chi}}^*(x)\boldsymbol{\tau}_i\boldsymbol{\chi}(x) = \exp\left[\mathrm{i}\varphi_0(x) + \mathrm{i}\varphi_3(x)\right]\left[\varphi_1(x) \times + \varphi_2(x) \times\right]\tilde{\boldsymbol{\chi}}^*(x)\boldsymbol{\tau}_3\boldsymbol{\chi}(x)\,, \tag{2.14}$$

with $\varphi_{1,2}(x)$ the two Goldstone fields of the isospin transformations, and $\varphi_0(x)$ an additional Goldstone field connected with the additional asymmetry of the ground state as exhibited by the non-Hermitian condition (2.13) with regard to the F-phase transformation. The latter implies the possibility for a freezing of the fermion number F by appropriate dressings and offers the opportunity to generate the new quantum number Λ introduced ad hoc in (2.9).

I will not go into a detailed discussion of the calculation of the various 'bound' states of the constituent fields, which was extensively described and treated in earlier papers [9, 39–41]. To succeed in producing finite results for masses and coupling constants, an effective change (averaging over an essential singularity) of the quantization rule was necessary, involving a cutoff of the divergent 2-point function at small distances $|x| \leq \ell$ (universal length) and applying the New–Tamm–Dancoff approximation method for calculating the masses of the nucleons and mesons. The simplest mesons consisted of the isoscalar and isovector Lorentz pseudoscalars and Lorentz-vector mesons generated from the S-bound state of the urfield/anti-urfield pairs $\boldsymbol{\chi}^*\boldsymbol{\chi}$, $\boldsymbol{\chi}^*\boldsymbol{\tau}\boldsymbol{\chi}$, $\boldsymbol{\chi}^*\sigma_\mu\boldsymbol{\chi}$, $\boldsymbol{\chi}^*\sigma_\mu\boldsymbol{\tau}\boldsymbol{\chi}$ phenomenologically connected to the η, π, ω and ρ and this at a time (1958) when only the π mesons were known. The mass of the η meson was correctly predicted relative to the π mass. A calculation of the Sommerfeld fine-structure constant was also attempted [42].

Numerical results could, however, only be obtained by using the rather crude cutoff procedure which had the serious consequence that the metric of the quantum-mechanical Hilbert space was no longer positive definite (Nevanlinna space). Therefore states with negative norm (ghost states) in principle could not be avoided, leading to a non-unitary S-matrix and violation of the probability interpretation of the wave functions. Details and many references can be found in Heisenberg's book [47].

Many interesting attempts were made using various dressing mechanisms to construct effective fields and interactions which could simulate the fields of the SM, mainly concentrating on the electromagnetic–weak interactions, but also including the difficult task of obtaining some understanding of the strong interaction and its apparent color $SU(3)$ quality [43–46].

I will restrict myself here to considerations concerning the symmetry group structure. The main weakness of the Heisenberg–Pauli spinor theory seemed to be connected to overshooting the effective regularization of the interaction to the extent that it was treated in practical calculations like a super-renormalizable theory where a coupling constant of mass dimension rather than of length dimension provided the ad hoc assumed basic length scale. This caused difficulties in generating gauge-invariant interactions, in particular electrodynamics from 'bound state' considerations. However, gauge-invariant interactions proved to be of decisive importance in the SM. On the other hand, gauge invariance (i.e., invariance under x-dependent symmetry transformations) is directly linked to the equal importance of the

kinetic and the interaction term, indicating the absence of an explicit length parameter (universal length) at the outset and hence requiring an invariance under scale transformations. This requirement leads to an even more radical formulation of the RUQFT, which will be sketched in the next section.

2.3.3 Gauge-Invariant and General Spacetime-Invariant Nonlinear Spinor Theory

To establish a gauge-invariant pure nonlinear spinor theory in the context of the RUQFT approach requires that gauge fields should not be added ad hoc but rather generated from the constituent spinor field, the urfield. This necessarily means that no spacetime derivatives, and hence no kinetic term for the urfield can occur in the basic Lagrangian or the corresponding field equation. The Lagrangian should only contain an 'interaction' term, a purely nonlinear expression of the constituent fields. It should be appreciated at this point that, because of the anticommuting property (Grassmann algebra) of the operator fields $\boldsymbol{\chi}$ and $\boldsymbol{\chi}^*$, the number of their local products is extremely limited. The Pauli exclusion principle limits their local clustering, so to speak, and restricts them to just one mode at each x-point. Because of the 4 components of the field $\boldsymbol{\chi}$ and also 4 for their Hermitian conjugates $\boldsymbol{\chi}^*$, the maximum local product will be an octonic expression, which also exhibits the maximum symmetry. This suggests [48, 49] starting with the maximally symmetrical action

$$S = \int \mathrm{d}^4x\sqrt{-g}L(x) \,, \tag{2.15}$$

with the ultralocal Lagrangian (density)

$$\sqrt{-g}L(x) = -\frac{1}{N} : \boldsymbol{\chi}\boldsymbol{\chi}\boldsymbol{\chi}\boldsymbol{\chi}\boldsymbol{\chi}^*\boldsymbol{\chi}^*\boldsymbol{\chi}^*\boldsymbol{\chi}^* : (x) = -\frac{1}{24N} : \det(\boldsymbol{\chi}\boldsymbol{\chi}^*) : (x) \,, \tag{2.16}$$

constructed from the 4-component, anticommuting, non-Hermitian isospinor–spinor field $\boldsymbol{\chi}(x) \neq \boldsymbol{\chi}^*(x)$ ('urfield'):

$$\boldsymbol{\chi} \equiv \boldsymbol{\chi}_\alpha(x) \,, \qquad \alpha = 1, 2, 3, 4 \,, \tag{2.17}$$

where N is an appropriate normalization factor. Such a self-interaction term was already considered in 1977 by Heinrich Saller [50].

This Lagrangian is 'ultralocal' in a more extended sense than used earlier by Klauder [51], that it lives solely on separate spacetime points and hence, prima facie, cannot generate interaction but only self-action, i.e., no dialogues but merely monologues. This, however, only holds in a classical interpretation of the fields. In the case of quantum fields, the products of the quantum fields become singular as a consequence of the quantization condition. These singularities have to be subtracted from the local product by a

regularization procedure, a Wick product prescription, which is indicated by the (still undefined) double-dots : : . This implies an infinitesimal nonlocality which can generate effective derivative terms for various local operators constructed from $\boldsymbol{\chi}$ and $\boldsymbol{\chi}^*$ and therefore corresponding effective kinetic terms in the Lagrangian and propagators. In particular, in the case of spontaneous symmetry breaking, there should appear Goldstone modes with 'soft' kinetic terms which disappear in the limit of small distances.

To secure scale-invariance of the action S, the urfield must be given the mass (or inverse length) dimension

$$\dim \boldsymbol{\chi}(x) = \dim \boldsymbol{\chi}^*(x) = \frac{1}{2} \,, \tag{2.18}$$

i.e., transform under scale transformations as

$$\boldsymbol{\chi} \longrightarrow \mathrm{e}^{\eta/2}\boldsymbol{\chi} \,, \tag{2.19}$$

which is subcanonical as compared to the canonical dimension $\dim \psi = 3/2$ of physical spinor fields ψ corresponding to $\psi^*\psi$ being a 3-space density.

To specify the Wick finite-part product : : , we postulate for the point-split $\boldsymbol{\chi}(x_+)\boldsymbol{\chi}^*(x_-)$ product the formal Laurent expansion in terms of the (timelike) split-vector ξ (with $x_\pm = x \pm \xi/2$) and in accordance with the dimension assignment (2.18):

$$\boldsymbol{\chi}_\alpha(x_+)\boldsymbol{\chi}^*_{\dot\beta}(x_-) = \frac{\mathrm{i}}{2\pi}\bar N \frac{\xi_\mu}{\xi^2} h^\mu_{\alpha\dot\beta}(x) + V_{\alpha\dot\beta}(x,\xi) \,, \tag{2.20}$$

with the first term explicitly exhibiting the singular part $\sim \xi^{-1}$. This can also be expressed by the condition

$$h^\mu_{\alpha\dot\beta}(x) = \frac{1}{\bar N} \lim_{\xi\to 0} \xi^\mu \boldsymbol{\chi}_\alpha(x_+)\boldsymbol{\chi}^*_{\dot\beta}(x_-) \neq 0 \,. \tag{2.21}$$

Hence the finite, ξ-independent Lagrangian (2.16) can now be cast into the more transparent form

$$\begin{aligned} \sqrt{-g}L(x) &= -\frac{1}{N} \lim_{\xi\to 0} X(x_+)X^*(x_-) \\ &= -\frac{1}{24\times 8N} \lim_{\xi\to 0} \partial^2_\xi \partial^2_\xi \xi^2 \xi^2 [X(x_+)X^*(x_-)] \,. \end{aligned} \tag{2.22}$$

The action (2.15) is formally invariant under the huge 47-parameter or respectively, the x-dependent parameter function ('local' or gauge) symmetry group

$$G_{\max} = \underset{\text{hybrid}}{D(1,\mathrm{loc})} \otimes \underset{\text{external}}{SL(4,R,\mathrm{loc})} \otimes \underset{\text{internal}}{U(1,\mathrm{loc})} \otimes SL(4,C,\mathrm{loc}) \,, \tag{2.23}$$

which contains:

- the external volume-conserving 15-parameter special linear transformations in the $d = 4$-dimensional spacetime manifold corresponding, in the x-dependent form, to the volume-conserving transformations of general relativity;
- the internal 1-parameter phase transformation for a fermion number F and the 30-parameter linear complex transformations of the $2n = 4$ spinor components (n = number of Weyl fields), ultimately interpreted as isospinor–spinor components;
- the 1-parameter hybrid internal–external dilations acting on the coordinate differentials as well as on the urfields according to the specification (2.19) under the condition $d = 2n$.

The Lagrangian even exhibits other symmetries. We can establish an invariance under spacetime translations and the general 'local' conformal transformations [50, 52, 53] and also an $N = 2$ supersymmetry [54].

The invariance of the action under the corresponding extended x-dependent transformations is, of course, a consequence of the formal complete separateness of all spacetime points. It is only because of the 'softening' of the 'ultra-locality' by the : : regularization prescription connected with the quantum character of the urfields without destroying this local invariance that the local invariance with regard to the internal groups now acquires the non-trivial meaning of a 'gauge' group. This implies that all derivatives which are effectively generated from the 'softening' of the nonlinear term now appear automatically as 'covariant' derivatives involving compensating vector gauge fields, which must be constructs from the urfields. In particular, we immediately realize [55] that the finite-part isovector–vector field

$$\boldsymbol{A}_\mu(x) = \frac{1}{2} : \boldsymbol{\chi}^* \sigma_\mu \boldsymbol{\tau} \boldsymbol{\chi} : (x) \,, \tag{2.24}$$

under x-dependent isospin transformations

$$\boldsymbol{\chi}(x) \longrightarrow \mathrm{e}^{\boldsymbol{\tau}\cdot\boldsymbol{\varphi}(x)/2}\boldsymbol{\chi}(x) = \left[1 + \frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{\varphi}(x) + \cdots\right]\boldsymbol{\chi}(x) \,, \tag{2.25}$$

automatically shows the inhomogeneous behavior

$$\boldsymbol{A}_\mu(x) \to \mathrm{e}^{\boldsymbol{\varphi}(x)\times}\boldsymbol{A}_\mu(x) + \partial_\mu\boldsymbol{\varphi}(x) = \boldsymbol{A}_\mu(x) + \boldsymbol{\varphi}(x)\times\boldsymbol{A}_\mu(x) + \partial_\mu\boldsymbol{\varphi}(x) + \cdots \,, \tag{2.26}$$

which always occurs in the combination of a covariant derivative

$$D_\mu \equiv \partial_\mu + \frac{\mathrm{i}}{2}\boldsymbol{\tau}\cdot\boldsymbol{A}_\mu \,. \tag{2.27}$$

However, the huge symmetry group (2.23) is not realized for the quantum field theory because of the smaller symmetry of the quantization condition of the fields under the following requirement for the anticommutator (suppressing a unit isospin matrix in the notation):

$$\lim_{\xi\to 0}\{\boldsymbol{\chi}_\alpha(x_+)\boldsymbol{\chi}^*_{\dot\beta}(x_-)\} = -\mathrm{Im}\frac{\bar N}{2\pi}\frac{\xi_\mu}{\xi^2-\epsilon(\xi^0)}h^\mu_{\alpha\dot\beta}(x)$$
$$= \bar N h^\mu_{\alpha\dot\beta}(x)\xi_\mu\epsilon(\xi^0)\delta(\xi^2)$$
$$= -\bar N|\xi_0|\delta(\xi_0^2) = \text{const.} \tag{2.28}$$

This quantization condition can also be phrased as a condition for the ground state, viz.,

$$\langle\Omega \mid h^\mu_{\alpha\dot\beta}(x) \mid \Omega\rangle = \delta^\mu_m\bar\sigma^m_{\alpha\dot\beta}\ . \tag{2.29}$$

This states that $h(x)^\mu_{\alpha\dot\beta}$ is essentially the mixed tetrad tensor related to the metric tensor in the usual way, and is required to contain the Minkowski flat space metric as exhibited by the most singular part of (2.28).

The quantization condition (2.28) induces a tremendous breaking of the maximum group (2.23) down to the 11-parameter stability group of the (assumed) translationally invariant ground state:

$$G_\Omega = \underset{\text{hybrid}}{D(1)} \quad \otimes \underset{\text{hybrid Lorentz}}{SO(3,1)} \quad \otimes \underset{\text{internal}}{U(1)\otimes SU(2)}\ , \tag{2.30}$$

giving rise to $47-11=36$ Goldstone modes connected with the 36-parameter coset

$$\frac{G}{G_\Omega} = \frac{SL(4,R)}{SO(3,1)}\otimes\frac{SL(4,C)}{SL(2,C)}\otimes SU(2)\ . \tag{2.31}$$

In contrast to the usual examples of a condensate caused by a Higgs boson field, no mass scale is established in our case on this first level of symmetry-breaking. It is rather the Planck constant $\hbar$ (not explicitly written) that is responsible. The connection between spacetime properties and spin degrees of freedom, investigated by many [56–58], is thus enforced by quantization; or the other way around, quantization provides the foundation for such a parameter space.

It is actually interesting to observe that the symmetry breakdown condition for the mixed tensor $h^\mu_{\alpha\dot\beta}$ enforces a fixed 'spin–orbit' coupling, which is the 4-dimensional analogue of a $J=2$ (S-triplet/P-wave). It imprints a Minkowski metric η_{ik}, natural to the spin algebra $(\sigma_i\bar\sigma_k)$, on the spacetime manifold. The 9 Goldstone excitations of this condensate characterize local deviations from the flat Minkowski metric and relate to the 9 (volume conserving) degrees of freedom of the usual 10 components of the gravitational field or the metric tensor with fixed determinant. Hence the ground state has some similarity with the B-phase of supercooled ^{3}He [59], in which a $^2P_2$ configuration with total spin $J=2$ occurs in such a way that, in the corresponding spontaneous breakdown of the orbital and spin symmetry (SBSOS), only the relative orientation of orbital and spin direction gets fixed

(pseudo-isotropy). The excitations of the relative orbital-spin (Goldstone) vibrating modes, the spin-nematic waves of the condensate, correspond to the gravitational degrees of freedom in our case.

The 21 Goldstone modes connected with the broken internal symmetries related to the coset $SL(4,C)/SU(2)$ will not really show up because they are absorbed by the corresponding vector gauge fields (Anderson–Higgs mechanism) [60,61] which are also generated, and will eventually give rise to massive vector particles.

In addition to this 'hard' symmetry breaking, we can imagine additional 'soft' breakings taking into account additional terms of the operators in a ξ-expansion $\sim m^2\xi\xi$, $m^4\xi\xi\xi\xi$, etc. In analogy with the usual symmetry-breaking mechanism, e.g., in the case of the phase transformation symmetry $U(1)$ or the isospin symmetry $SU(2)$ as indicated in (2.13), they would then produce corresponding kinetic terms. It should be noticed in this context that the scalar bilinear forms $:\boldsymbol{\chi}(x)\boldsymbol{\chi}(x):$ and $:\boldsymbol{\chi}^*(x)\boldsymbol{\chi}^*(x):$ do not actually require : : regularization because the anticommutator of like fields vanishes.

In a similar way additional terms $\sim M^2\xi\xi$ on the right-hand side of (2.20) in $V_{\alpha\dot{\beta}}(x,\xi)$ would give rise to a general covariant kinetic term for the gravitational field $\sim M^2$ with the important consequence that, in comparison with the interaction term, a gravitational coupling constant $\sim 1/M^2 \sim \ell^2_{\text{Planck}}$ would naturally occur. This means that M should be identified with the huge gravitational mass

$$M = \ell^{-1}_{\text{Planck}} = 1.22 \times 10^{19}\ \text{GeV}\ . \tag{2.32}$$

The explicit evaluation of the Lagrangian (2.22) produces the various derivative forms connected to kinetic terms and derivative couplings. They are rather numerous and are given elsewhere [49]. The leading term is a third-derivative expression,

$$L(x) = \frac{2\pi^2}{\bar{N}} :\boldsymbol{\chi}^* \frac{\mathrm{i}}{2}^3 \sigma \overleftrightarrow{\partial}\overleftrightarrow{\partial^2}\boldsymbol{\chi}: (x) + \cdots\ , \tag{2.33}$$

with the normalization factor $N = \bar{N}^4/96\pi^2$, where we have suppressed the gauge fields which augment the derivatives to the covariant expressions.

The corresponding Green's function or propagator depicts a 'double pole' in momentum space instead of the common single pole for a particle:

$$G(p) \sim \bar{N}\frac{\bar{\sigma}\cdot p}{(p^2)^2}\ , \tag{2.34}$$

which correctly reflects the anticommutator rules for a fermion of subcanonical dimension 1/2. The double pole describes a 'dipole ghost', treated extensively by Heisenberg in 1957 [20] and refers to unphysical states of zero norm and therefore zero probability.

If a softer mass term $\sim M^2$ appears in the expansion of $V[x,\xi]$ in (2.20) then this double pole may be pulled apart to positive and negative norm contributions:

$$G(p) \sim \bar{N}\frac{\bar{\sigma}\cdot p}{p^2(p^2 \pm M^2)} = \pm\bar{N}\frac{\bar{\sigma}\cdot p}{M^2}\left(\frac{1}{p^2} - \frac{1}{p^2 \pm M^2}\right) , \tag{2.35}$$

which looks like a regular neutrino propagator with a momentum cutoff at the mass M, e.g., like the huge gravitational mass (2.32) mentioned above. If the minus sign occurs, the second term will correspond to a negative norm (ghost) fermion of mass M (if $\bar{N} = -1$ is chosen).

I will not go into further details here. It can be demonstrated [49] that essentially all the effective local fields of the Standard Model can be generated from the urfield. This is easily imagined if we consider different groupings of the octonic interaction:

$$\begin{array}{ll} \text{Interaction term} & \boldsymbol{\chi}^*\boldsymbol{\chi}^*\boldsymbol{\chi}^*\boldsymbol{\chi}^*\boldsymbol{\chi}\boldsymbol{\chi}\boldsymbol{\chi}\boldsymbol{\chi} \\ \text{Higgs analog} & \boldsymbol{\Phi}^*\boldsymbol{\Phi}^*\boldsymbol{\Phi}\boldsymbol{\Phi} \sim (\boldsymbol{\chi}^*\boldsymbol{\chi}^*)(\boldsymbol{\chi}^*\boldsymbol{\chi}^*)(\boldsymbol{\chi}\boldsymbol{\chi})(\boldsymbol{\chi}\boldsymbol{\chi}) \\ \text{Fermion–gauge} & \boldsymbol{\psi}^*\boldsymbol{A}\boldsymbol{\psi} \sim (\boldsymbol{\chi}^*\boldsymbol{\chi}^*\boldsymbol{\chi}^*)(\boldsymbol{\chi}^*\boldsymbol{\chi})(\boldsymbol{\chi}\boldsymbol{\chi}\boldsymbol{\chi}) \\ \text{Higgs–gauge} & \boldsymbol{\Phi}^*\boldsymbol{A}\boldsymbol{A}\boldsymbol{\Phi} \sim (\boldsymbol{\chi}^*\boldsymbol{\chi}^*)(\boldsymbol{\chi}^*\boldsymbol{\chi})(\boldsymbol{\chi}^*\boldsymbol{\chi})(\boldsymbol{\chi}\boldsymbol{\chi}) \end{array} \tag{2.36}$$

and the possible kinetic terms

$$\begin{array}{llll} \text{Fermions} & \boldsymbol{\chi}^*\partial\partial\partial\boldsymbol{\chi} & \boldsymbol{\psi}^*\partial\boldsymbol{\psi} & M^2\boldsymbol{\chi}^*\partial\boldsymbol{\chi} \\ \text{Higgs} & & \partial\boldsymbol{\Phi}^*\partial\boldsymbol{\Phi} & m^2\boldsymbol{\Phi}^*\boldsymbol{\Phi} \\ \text{Gauge} & \partial\boldsymbol{A}\partial\boldsymbol{A} & \boldsymbol{A}\boldsymbol{A}\partial\boldsymbol{A} \;\; \boldsymbol{A}\boldsymbol{A}\boldsymbol{A}\boldsymbol{A} & \\ \text{Goldstone} & m^2\partial\varphi\partial\varphi & M^2\partial h\partial h & \end{array} \tag{2.37}$$

with the effective fields

$$\begin{array}{llll} \text{Canonical fermions} & \dim = 3/2 & \text{left} : \boldsymbol{\psi} = \boldsymbol{\chi}\boldsymbol{\chi}\boldsymbol{\chi} & \text{right} : \boldsymbol{\psi}' = \boldsymbol{\chi}^*\boldsymbol{\chi}\boldsymbol{\chi} \\ \text{Canonical bosons} & \dim = 1 & \boldsymbol{\Phi} = \boldsymbol{\chi}^*\boldsymbol{\chi} & \boldsymbol{A} = \boldsymbol{\chi}^*\boldsymbol{\chi} \\ \text{Goldstone fields} & \dim = 0 & \varphi_\pm \quad v & g_{\mu\nu} \quad h^{\mu}_{\alpha\dot{\beta}} \end{array} \tag{2.38}$$

An important feature are the Goldstone dressings arising from the combined symmetry breakdown of the isospin group and the fermion number phase transformation. The Goldstone degrees of freedom can be formally accentuated by writing the urfield as

$$\begin{aligned} \boldsymbol{\chi}_\alpha(x) &= \exp\left\{\frac{\mathrm{i}}{2}[\varphi_+(x)\tau^- + \varphi_-(x)\tau^+]\right\}\exp\left\{\frac{\mathrm{i}}{2}[\varphi_0(x) + \varphi_3\tau^3(x)]\right\}\underset{-}{\boldsymbol{\chi}}(x) \\ &= \boldsymbol{s}^3_\alpha(x)v_3(x)\underset{-}{\boldsymbol{\chi}}(x) . \end{aligned} \tag{2.39}$$

Here $\boldsymbol{s}^3_\alpha(\varphi_\pm)$ is a transmutator $I_{1,2,3} \to I_3 = Y$, freezing the isospin into the third direction, the hypercharge Y (now like an isoscalar) with only local deviations as expressed by the Goldstone modes $\varphi_\pm(x)$. Furthermore, $v_3(\varphi_0)$ is a transmutator tying the I_3 to the fermion number to enforce $F - I_3 = 0$.

Formally $s(x)$ acts like a local isospinor with an additional frozen charge $Y = 1/2$. It has the properties of Wentzel's spurion [34,38]. $v(x)$ is a local isocalar field with $F = -Y = 1/2$. As a consequence there will be numerous additional effective local fields resulting from different dressings of the constituent field constructs. The electric charge Q of the (asymptotically emerging) dressed particles will only result from their isospin degree of freedom:

$$Q = (I_3)_{\text{total}} = (I_3)_{\text{field}} + (I_3)_{\text{dressing}} \,, \tag{2.40}$$

and the same holds for the relevant fermion number

$$F = F_{\text{total}} = (F)_{\text{field}} + (F)_{\text{dressing}} \,. \tag{2.41}$$

Different dressings, involving an infinite number of Goldstone modes, should represent a highly effective barrier for transitions and hence should give rise to a large number of fermion-number-type conservation laws. As mentioned earlier, this may not only be the key to obtaining independent conservation for baryon and lepton numbers, but it may also offer a hint for the existence of the different flavor families. The left-handed leptons of canonical dim = 3/2, the neutrino–electron doublet, may be straightforwardly constructed as a dressed triple field

$$\boldsymbol{\psi}_{\text{L}} \sim v\boldsymbol{\chi\chi\chi} \,, \qquad F = 2 \,, \qquad Y = -\frac{1}{2} \,, \tag{2.42}$$

and a right-handed isoscalar field arises by involving an anti-field

$$\psi'_{\text{L}} \sim v^3 \boldsymbol{s\chi\chi\chi}^* \,, \qquad F = 2 \,, \qquad Y = -\frac{1}{2} \,, \tag{2.43}$$

offering with (2.42) the possibility to establish a massive electron. The right-handed nucleons, the proton–neutron doublet, on the other hand, may have the structure

$$\boldsymbol{\psi}_N \sim v^*\boldsymbol{\chi\chi\chi} \,, \qquad F = 1 \,, \qquad Y = +\frac{1}{2} \,. \tag{2.44}$$

They may be regarded as a composition of three quarks $q \sim (v^*)^{1/3}\boldsymbol{\chi}$, a very artificial construction. It would rather suggest dealing instead with a 4-component system with three fermions, each of only half a charge being embedded into a fourth partner, a smeared out ($F = -1/2$, $Q = +1/2$) Goldstone halo, or an extended soliton-type bag. This may all sound strange, but there is at this stage no indicator for establishing an $SU(3)$ color symmetry. Hence, this still represents a serious weakness in the present theory. Many attempts were made to remedy the situation but none proved satisfactory, in particular regarding the apparent high validity of $SU(3)$ color invariance demonstrated by experiments. Nevertheless, I believe we should not give up searching for an adequate solution of the color aspect within our framework.

To my mind, the dressing mechanism is very important and has shown many interesting features [43–46] which have hardly drawn much attention up to now and hence have not been looked at in more detail by others. A comparison with atomic physics, for example, as indicated by the boron example (Figs. 2.1 and 2.2), shows very clearly how in many-fermion systems the energy term system (analogous to the particle spectrum) exhibits approximately valid new symmetries, which are very remote from the symmetries of the underlying dynamics of the constituents. In particular $SU(n)$ symmetries, with n the number of the coexisting electrons in a shell or subshell, play an important role because of their antisymmetrization (permutation group).

The importance of the dressing aspect related to the phenomenon of spontaneous symmetry breakdown and its formulation by Bose–Einstein condensation with the occurrence of zero-mass Goldstone modes and hence the possibility of small energy deformations has recently drawn a lot of attention [63] connected with new quantum technologies, and specifically regarding quantum computers, and it has even reached the popular press [62]. Indeed, this may be another indication that, beyond the appearance of the gapless excitations in a Bose–Einstein condensate in terms of the massless Goldstone modes, there will be, in general, a high spatial deformation instability or sensibility connected with the possibility of low-energy Goldstone clusters of finite extensions or, if interactions are appropriately taken into account, even of localized 'bound states' of such clusters. These may serve as traps for spinorial field configurations leading to soliton-type bags for fermions (dressings) as required to explain the spatial extension of hadrons without employing the usually assumed gluon fields. Such traps effectively suppress transitions between fermion configurations which otherwise appear to be allowed on the basis of non-gauge-type fermion number conservation. Hence the still mysterious high asymmetry of the universe regarding baryon number B and lepton number L, may not actually be related to a PC-violation but rather to an incorrect assignment of B and L to the basic (fermion number) $U(1)$ symmetry.

It is quite cumbersome to pull out the physically interesting aspects explicitly from the constituent field dynamics represented by the Lagrangian, as explicitly demonstrated in [48,49], and in particular to calculate numerical mass ratios and coupling constants explicitly. We should not be surprised that this 'dynamical map' [64,65] from local to asymptotic expressions or bare to dressed fields will be and must necessarily be very complicated. It has been demonstrated in simpler cases how this can be done in a very rough way, but better methods should certainly be developed. Phenomenologically more accessible formulations may be obtained by introducing effective local fields through appropriate constraints using the Lagrange multiplier approach.

Although the suggested formulation of our basic theory [66–68] may still be very far from being called a robust theory or even a theory at all, let me finish with some general remarks about how we might go even further in our

attempts to be radically quantum. This will be a highly speculative journey but may encourage new thinking and serve to open new vistas.

Our Lagrangian has still some ad hoc basic features which, from our radical point of view, require further probing. One is the introduction of a 4-dimensional spacetime as a parameter background for our description; the other is the reason for using a 4-component non-Hermitian spinor field instead of the simpler 2-component field, or even something else. There is no question that a spinor field has tremendous advantages over other fields to serve as a constituent field. This is obvious (half-spin, anticommutativity). The spacetime continuum we have chosen as background with an approximately Minkowskian pseudo-Euclidean metric structure gives the 'time' dimension a different role than the three 'space' dimensions. But this background is simply a continuous 4-fold label (similar to the index α of the spinor) for the 'octonic spinor field cluster' which defines the Lagrangian, or better, the Lagrangian density or the basic 'building block'. It is just an enumeration of 'more of the same' and, in fact, an infinite number of these clusters. The action is just the total sum or, if really considered continuous, an integral over all these 'octonic stars'. The 'time' plays an important role, because we use it to give meaning to an order in the definition of the Wick product. The Wick product prescription is based on a time-ordering definition for the fields regarded as operators connecting the time sequence with the consecutive multiplication from the left, defining a time arrow (the 'earlier' is always to the right of the 'later'). The operator character of the spinor is essential and is reflected in the anticommutativity of its products. We can characterize this Grassmann property by step operators $a, a^\dagger$,

$$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \qquad a^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \tag{2.45}$$

satisfying the anticommutation rules

$$\{a, a\} = \{a^\dagger, a^\dagger\} = 0 , \qquad \{a, a^\dagger\} = 1 . \tag{2.46}$$

These step operators describe possible changes in a 2-state system, e.g., of 'nonexistence' and 'existence'. Then with the time-arrow interpretation it gets the meaning

$$\begin{aligned} a^\dagger &= (\text{not to be}) \longrightarrow (\text{to be}) = \text{creation operator} , \\ a \ &= (\text{not to be}) \longleftarrow (\text{to be}) = \text{annihilation operator} . \end{aligned} \tag{2.47}$$

I consider this relationship to be rather profound in the sense that the operation obtains the meaning of a directed process in a parameter sequence which we call time.

The 4-component constituent spinor $\boldsymbol{\chi}_\alpha$ (on one x-point) now relates to four independent pairs of such step operators $a, a^\dagger$. The question now arises: why do we need a doubled Weyl spinor $\boldsymbol{\chi}$ and not simply χ? What is the reason for the isospin?

In our formulation, the isospin definitely played a secondary role in comparison with the spin, which is closely connected to the spacetime metric. But the isospin doubling nonetheless proved essential in defining an octonic product. Maximally, a single Weyl spinor allows only a quartic product as used in the Heisenberg–Pauli equation (2.6). But this is not actually the case because $\boldsymbol{\chi}_\alpha$ is non-Hermitian and hence $\boldsymbol{\chi}^*_{\dot\beta}$ can be connected with other pairs $b, b^\dagger$. This can be readily seen if we restrict ourselves to a single Weyl spinor χ_α, which as usual allows the introduction of a_1, a_2, b_1, b_2 and $a_1^\dagger, a_1^\dagger, b_1^\dagger, b_2^\dagger$ with indices 1, 2 referring to the spin up/down components of the Weyl spinor (helicity). This does indeed admit a non-vanishing octonic product:

$$a_1^\dagger a_2^\dagger b_1^\dagger b_2^\dagger a_1 a_2 b_1 b_2 \ . \tag{2.48}$$

Why does this work? We have dissociated the usual connection between the fermion number F and helicity. Perhaps the old Pauli definition of the 4-component spinor as a Majorana–Dirac spinor may after all be more appropriate, with the important addition now, that the Majorana condition $\psi = \gamma_5 \psi^{\mathrm{C}}$ no longer holds for the operators. This would imply that the basic 8-parameter symmetry group

$$GL(2,C) = D(1)(\eta_0) \otimes \frac{SL(2,C)}{SU(2)}(\eta) \otimes U(1)(\alpha_0) \otimes SU(2)(\alpha) \tag{2.49}$$

could perhaps suffice.

The orbits of the non-compact dilatation group and the coset depending on the parameters (η_0, η) applied to a positive or negative timelike vector fill the forward and backward light cones, respectively. For fixed η_0, we obtain the 3-dimensional space on a hyperbola. For $\eta_0 \to -\infty$, this shrinks back to the light cone. Only in this limit are the positive and negative light cones connected at the origin. This suggests identifying the ad hoc introduced spacetime continuum on which the octonic spinor stars are spread out with the 4-parameter space of the non-compact part of the $GL(2,C)$ group. The octonic spinor star consists of four annihilation beams entering from the backward light cone, and four creation beams emerging into the forward light cone, with both 4-beams not quite touching at the origin because of the regularization of the self-action.

It is tempting to connect the evolution of the cosmos with a steady increase in the dilatation parameter η_0, starting at minus infinity at the origin and sweeping step by step over the whole future cone, enlarging the number of spinor stars along the 3-space hyerbola, which will look more and more like our familiar infinite spacetime continuum. The Planck length ℓ_{Planck} may perhaps be connected with the 'minimum time' of the creation and annihilation processes inherent in the basic step operators, i.e., if interpreted according to

$$a \cdot a^\dagger + a^\dagger \cdot a \Longrightarrow \frac{1}{\sqrt{\ell}} \left[a\left(+\frac{\ell}{2}\right) a^\dagger\left(-\frac{\ell}{2}\right) + a^\dagger\left(+\frac{\ell}{2}\right) a\left(-\frac{\ell}{2}\right) \right] \ , \tag{2.50}$$

where the factor in front prepares for $\dim\boldsymbol{\chi} = 1/2$ under dilatations. However, because of the scale invariance of the dynamics, the Planck length will only show up at longer distances in the form of logarithmic mass terms, or in softer terms connected with spontaneous symmetry breakdowns.

In a way, the cosmos would resemble a huge, continuously growing, parallel closely-linked computer system with software based on the 8-fold general linear transformations in a 2-dimensional complex space $GL(2, C)$ instead of the (0,1)-bit of our present computers. However, because of the basically creative elements and the infinitely open logic, the quantum cosmos, in stark contrast to our fully determined computer, would be essentially open to the future, and hence would correspond more closely to what in our limited meso-world we experience as being 'fully alive'.

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3 Quantum Physics as a Science of Information

Časlav Brukner and Anton Zeilinger

There are at least three different ways in which quantum physics is connected with the concept of information. One is the relationship between quantum interference and knowledge. This was at the very heart of the early debates concerning the meaning of quantum mechanics, most notably the Bohr–Einstein dialogue [1]. That debate concerned the problem which occurred when quantum mechanics came up against the hitherto accepted notion that physics must describe reality as directly as possible and in an unambiguous and complete way. The debate was resolved by the Copenhagen interpretation in the most radical, conceptually challenging and foresightful manner, although for many physicists today, the Copenhagen interpretation is still conceptually unacceptable.

The second connection between quantum physics and information was the discovery in the early 1990s that quantum concepts could be used for communication and for processing information in completely novel ways. These include such topics as quantum cryptography, quantum teleportation and quantum computation [2].

The third connection between quantum physics and information has been emerging gradually over the last few years with the conceptual groundwork for this connection going back to the works of von Weizsaecker [3] and Wheeler [4]. It is the notion that information is the basic concept of quantum physics itself. That is, quantum physics is only indirectly a science of reality but more immediately a science of knowledge.

The present paper will touch upon all three connections between quantum physics and information.

3.1 Information and Quantum Interference

The connection between quantum interference and information is best illustrated by the double slit experiment (Fig. 3.1). This experiment already poses the challenging question: Does quantum mechanics describe reality or information? If we briefly consider the experiment with electrons – under what conditions do interference fringes arise at the observation plane?

Such fringes can easily be understood on the basis of interference of waves passing through both slits. Yet, as soon as we perform the experiment with

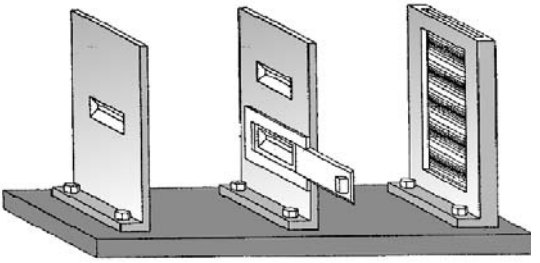

Fig. 3.1. Double slit experiment. The figure is taken from Bohr's article "Discussion with Einstein on epistemological problems in atomic physics" [1]. There Bohr writes: "This point is of great logical consequence, since it is only the circumstance that we are presented with a choice of either tracing the path of a particle or observing interference effects, which allows us to escape from the paradoxical necessity of concluding that the behaviour of an electron or a photon should depend on the presence of a slit in the diaphragm through which it could be proved not to pass. We have here to do with a typical example of how the complementary phenomena appear under mutually exclusive experimental arrangements and are just faced with the impossibility, in the analysis of quantum effects, of drawing any sharp separation between an independent behaviour of atomic objects and their interaction with the measuring instruments which serve to define the conditions under which the phenomena occur"

single particles, as has been done with photons, electrons, neutrons, atoms and molecules, the question arises: How does an individual particle which, one would naturally expect, has to pass through either slit, know whether or not the other slit is open? Richard Feynman [5] wrote for the double-slit experiment: "In reality, it contains the only mystery." The modern Copenhagen way to talk about these questions is to assume that it only makes sense to talk about a property of a system if one actually cares to determine it or if at least the possibility for determining it exists. Or, in an even more modern way, the interference fringes arise if and only if there is no possibility, not even in principle, to determine which path the particle took. And, most importantly, it is not relevant whether or not we care to take note of that information. All that is necessary is whether or not the information is present somewhere in the universe. Only if such information is not present do interference fringes occur.

Indeed, the most interesting situations arise if the path information is present at some point in time, but deleted or erased in an irrevocable way later on. Then, as soon as that information is irrevocably deleted, the interference fringes can occur again. Here, it is important to note that the mere diffusion of the information into larger systems, maybe even as large as the whole universe, is not enough to destroy the information. As long as it is there, no matter how well hidden or how dispersed, the interference fringes cannot occur.

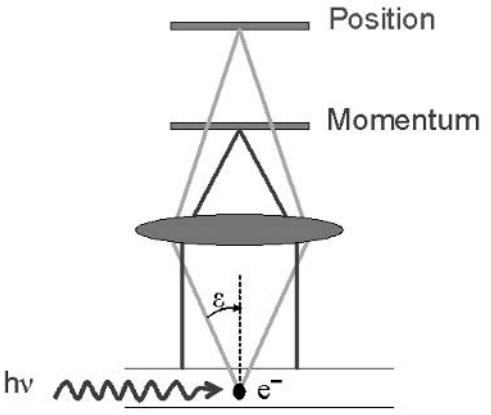

Fig. 3.2. Two mutually complementary arrangements of the Heisenberg microscope. A photon is scattered from an electron and then enters the Heisenberg microscope. If a detector (or an observation screen) is placed in the imaging plane of the Heisenberg microscope lens, it can reveal the path the electron takes through the slit assembly, which therefore cannot show an interference pattern, assuming that it passes through an interferometer. On the other hand, if the detector is placed in the focal plane of the lens, it projects the state of the electron into a momentum eigenstate which cannot reveal any position information and therefore no information about which slit the electron has passed through. Interference fringes may thus occur

This feature is most interestingly shown in a gedanken experiment where one combines the famous Heisenberg microscope with an electron double slit experiment.

We consider an electron interferometer where we may, if we decide to do so, determine the path the electron takes. This is done by scattering photons from the electrons passing through the double slit assembly. Clearly, these photons can be used to determine the path the electrons take by finding out in which slit they have been scattered (assuming that their wavelength is short enough). The simplest way to determine that position of scattering is to use the Heisenberg microscope as shown in Fig. 3.2. One can simply put a position-sensitive photon detector into the image plane and then, depending on where one observes the photon, one knows the path taken by the electrons. Therefore, no interference fringes can occur in that situation.

One might say now that one could simply not determine the position by not putting a photon detector into the image plane. Then one does not obtain information about the path taken, and one might be tempted to argue that interference fringes should occur. Yet the scattered photons nevertheless carry away the information about where they have been scattered and the path taken could be determined at an arbitrary time. Therefore, even if one does not care to read out this information, interference fringes should not arise.

Indeed, one could imagine that someone in a distant galaxy, equipped with very advanced technology, collects enough of the probability wave of the photon scattered and is thus able to determine the path taken. Therefore, even if one does not look at the scattered photon, no interference fringes should arise for the electron as long as the photon carries the path information.

In order to obtain interference fringes, one has to erase the information carried by the photon in an irrevocable way. That can best be done by detecting the photon, not in the image plane, but in the focal plane of the lens. Recalling the basic concept of Fourier optics, we realize that a point in the focal plane of a lens corresponds to an incoming momentum (or direction) on the other side of the lens. Thus it follows that registration of the photon in the focal plane projects the state of the scattered photon onto a momentum eigenstate which does not contain any position information. Therefore, once the photon is registered in the focal plane, all position information is gone and the corresponding electron interferes with itself.

This experiment has actually been realized, not using an electron and a photon, but using two photons exploiting the notion of entanglement [6,7]. In this experiment (Fig. 3.3), one creates pairs of momentum entangled photons. One of the two photons plays the role of the electron and passes through a double slit assembly. That photon is registered behind the double slit at the double slit detector. The other photon plays the role of the scattered photon in the Heisenberg microscope experiment. It passes through the Heisenberg lens and then to the Heisenberg detector. Because of the strong entanglement between the two photons, the photon passing through the double slit does not show any interference pattern. In fact, the photon passing through the Heisenberg lens can be used to determine the path taken by the photon passing through the double slit. This is done by placing the detector in the image plane of the lens.

Alternatively, if one places the Heisenberg detector in the focal plane, the incoming photon and therefore also the entangled photon are both projected onto momentum eigenstates and the double slit interference fringes arise for photons observed in coincidence with a registration at the Heisenberg detector (see Figs. 3.3 and 3.4).

In this experiment, we also notice an interesting and elegant feature, viz., the low count rate. The peak count rate in the double slit pattern in Fig. 3.4 is about 120 photons in 60 seconds. This means that it is absolutely beyond doubt that the interference pattern is built up individual photon by individual photon.

These experiments can be seen as a confirmation of the viewpoint that it does not make sense to assign any property to a physical system irrespective of observation. In our case, the property as to whether the photon passing the double slit assembly can be seen as a particle or as a wave depends on what happens to the first photon. And this may actually occur at a time after the photon passing through the double slit assembly has already been registered!

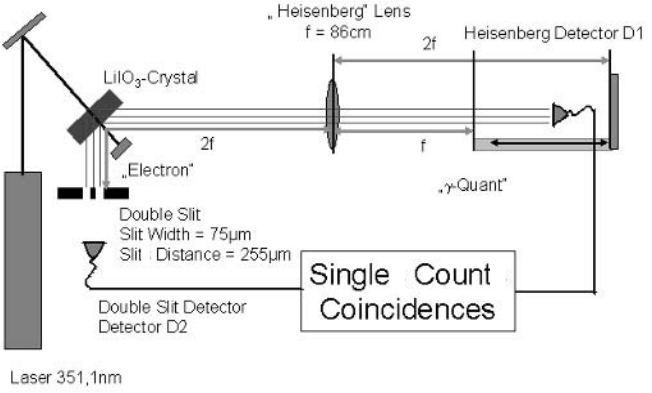

Fig. 3.3. Double slit experiment for a photon of an entangled pair [6, 7]. A pair of momentum-entangled photons is produced in the crystal by type-I parametric down-conversion. One of the photons enters the Heisenberg microscope and is detected by the Heisenberg detector placed behind the Heisenberg lens. (It plays the role of the γ-quantum in the standard Heisenberg microscope experiment.) The other photon enters the double slit assembly and is detected by the double slit detector. (It plays the role of the electron.) If the Heisenberg detector is placed in the imaging plane of the lens, it can reveal the path the other photon takes through the slit assembly, which therefore cannot show interference. Alternatively, if the Heisenberg detector is placed in the focal plane of the lens, it projects the state of the other photon into a momentum eigenstate which cannot reveal any information about the slit the photon passes through. This photon therefore exhibits an interference pattern in coincidence with the registration of the other photon in the focal plane of the Heisenberg lens

One might view this as a nice corroboration of Niels Bohr's famous dictum: "No phenomenon is a phenomenon unless it is an observed phenomenon."

These experiments also shed interesting light on the role of the observer with respect to reality. We note that it is the experimentalist who chooses the apparatus. The experimentalist, in our case Birgit, decides whether to put the detector into the focal plane or, say, into the image plane. That way, she determines which property of the system, wave or particle, can be reality. We might thus conclude that the experimentalist choosing the apparatus determines which physical quantity, i.e., quality, can be reality. In that sense, the experimentalist's choice is constitutive of the universe. However, the specific outcome here, which of the two slits the particle passes through in one case or where on the observation plane it arrives in the other, cannot be influenced by her. That way, Nature avoids complete controllability by the observer.

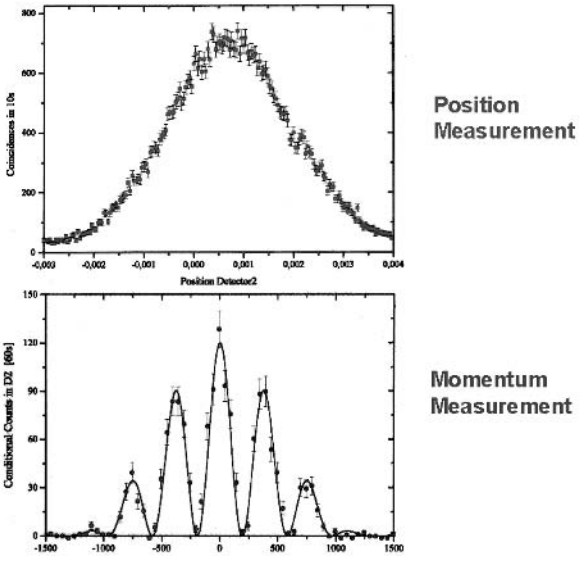

Fig. 3.4. Two mutually exclusive patterns registered by the double slit detector placed behind the double slit assembly (Fig. 3.3) as a function of its lateral position. The graphs show counts registered by that detector in coincidence with the registration at the Heisenberg detector if it is placed in the imaging plane of the lens (*upper*) and if it is placed in the focal plane of the lens (*lower*). Only in the latter case do the counts exhibit an interference pattern as the observation at the Heisenberg detector does not reveal the path the photon takes through the double slit assembly. Note the low intensity which indicates that the interference pattern is built up by individual photons, one at a time

3.2 Towards a Quantum Information Technology

Unexpectedly for many, experiments motivated by fundamental and philosophical concerns have led to novel concepts for processing information. Quantum communication and quantum computation are the two areas where such new protocols in information technology have been developed over the last few years [2]. Interestingly, these new concepts and protocols rely on three fundamental notions. These are:

- the randomness of the individual measurement outcome,
- quantum complementarity,
- quantum entanglement.

A scheme which uses all three concepts together is entanglement-based quantum cryptography [8]. Let us therefore briefly discuss the essential point of the protocol without going into too much detail.

Let us assume that Alice and Bob share the entangled state of two qubits

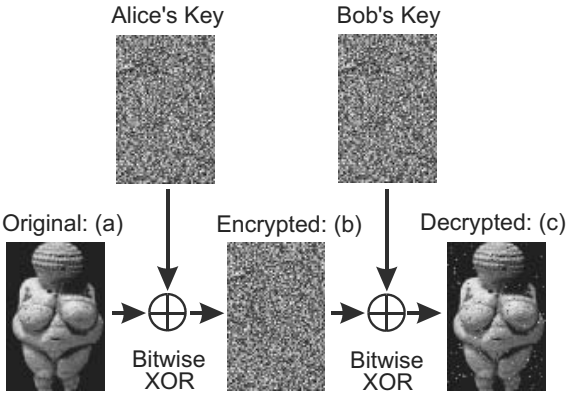

Fig. 3.5. Encryption of an image of the Venus of Willendorf in the experimental realization of quantum cryptography [11]. The image is encrypted by Alice via bitwise XOR operation with her key. She transmits the encrypted image (**b**) to Bob via the computer network. Bob decrypts the image with his key, resulting in (**c**), which shows only a few errors due to the remaining bit errors in the keys

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle_{\mathrm{A}}|0\rangle_{\mathrm{B}} + |1\rangle_{\mathrm{A}}|1\rangle_{\mathrm{B}}\big)\ . \tag{3.1}$$

Here a qubit is a two-state quantum system, the base states being denoted as $|0\rangle$ and $|1\rangle$ corresponding to the bit values 0 and 1, respectively. The state (3.1) is one of the maximally entangled (Bell) states [9]. The qubit A is held by Alice and qubit B by Bob after they have been produced somehow in the entangled state. It is now evident that, if Alice and Bob both perform measurements in the $\big\{|0\rangle, |1\rangle\big\}$ basis, they will obtain the same random result 0 or 1 on their qubits. Hence, after having measured many pairs, the two arrive at identical sequences of random numbers. These numbers can be used as keys to encode information. It has been known since Vernam [10] that such keys are secure under two conditions: firstly, that they are used only once (one-time-pad) and secondly, that they are completely random. The procedure to detect an eavesdropper is to switch randomly between two bases, the computational one $\big\{|0\rangle, |1\rangle\big\}$ and the complementary one, often also referred to as conjugate:

$$|0'\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle - |1\rangle\big)\ , \quad |1'\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big)\ . \tag{3.2}$$

In the conjugate basis the entangled photon has the same mathematical form as (3.1). Alice and Bob will switch randomly and independently between the two bases. Evidently they obtain the same bit value whenever they happen to have the same basis. An eavesdropper, Eve, somewhere on the line has to guess which basis Alice and Bob chose. Clearly, her guess will fail frequently. On the other hand, if she tries to eavesdrop, e.g., by interacting her qubits with those of Alice and/or Bob, this will necessarily induce errors between

the results of Alice and Bob that can easily be detected by them. Therefore she can easily be detected by checking the errors established between the results of Alice and Bob. Finally, using the well-established procedure, Alice and Bob can both arrive at the identical and secure random bit sequence that can be used as a key to encrypt information. For more details and various protocols, we refer the reader to the literature [2].

The first experimental realization [11] used polarization-encoded qubits with the simple identification $|0\rangle = |H\rangle$ and $|1\rangle = |V\rangle$, where H and V denote a horizontally and a vertically polarized photon, respectively (see Fig. 3.5).

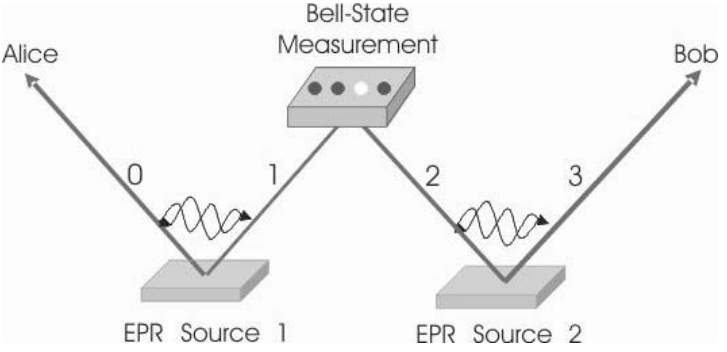

Fig. 3.6. Scheme for entanglement swapping, i.e., the teleportation of entanglement. Two pairs of entangled qubits 0–1 and 2–3 are produced by two Einstein–Podolsky–Rosen (EPR) sources. One qubit from each of the pairs is sent to two separated observers, say qubit 0 is sent to Alice and qubit 3 to Bob. The other qubits 1 and 2 from each pair become entangled through a Bell-state measurement, whereby qubits 0 and 3 also become entangled. This requires the entangled qubits 0 and 3 neither to come from a common source nor to have interacted in the past

Another protocol utilizing all three fundamental concepts, randomness, complementarity, and entanglement, is quantum teleportation [12,13], where one can transfer the quantum state of one system to another over arbitrary distances without physically transferring the system itself. The most interesting realization of quantum teleportation occurs when an entangled state itself is teleported, also called entanglement swapping [14,15]. In this experiment, as shown schematically in Fig. 3.6, one starts with two entangled pairs of qubits and performs a Bell-state measurement on one qubit from each pair. That way, the other two qubits, no matter how far they might be separated from each other, become entangled even though they share no common past. This protocol is conceptually very interesting, as it can be viewed as quantum teleportation of qubits which do not even have their own well-defined state. This is because an entangled qubit itself can only be described by a mixed density matrix. In a recent experiment [16], it has been possible to perform the Bell-state measurement with sufficient quality for the two outer, newly

entangled photons to become so highly entangled that the Bell inequality was violated. We might mention that such schemes may be of importance in future long-distance quantum communication protocols involving, for example, quantum repeaters [17].

There are many other applications of fundamental quantum concepts in new information technology protocols. These include, most notably, quantum computation, which is seen by many, including the present authors, as the future, albeit maybe long-term, of computation.

3.3 Quantum Physics as a Science of Information

The various debates about the conceptual significance of quantum mechanics can to a large extent be seen as a debate about what quantum physics refers to. Does it refer to reality directly or does it refer to (our) knowledge, and therefore to information? If quantum physics refers to reality, which reality is it? Is it the reality which appears to us, or is it a more complicated reality, like the one alluded to in the many-worlds interpretation?

We suggest that significant inspiration can be obtained from Niels Bohr, who, for example, according to Aage Petersen liked to say [18]: "There is no quantum world. There is only an abstract quantum physical description. It is wrong to think that the task of physics is to find out how Nature is. Physics concerns what we can say about Nature."

To us, it is thus suggestive that knowledge is the central concept of quantum physics. In modern language, knowledge can be equated with information. Therefore, one needs first a proper measure of information. One might be tempted to use Shannon's measure

$$I = -\sum_i p_i \log p_i \ , \tag{3.3}$$

where p_i is the probability of sign i occurring in a sequence. Yet it turns out that Shannon's measure is not adequate to describe the knowledge gained in an individual quantum experiment [19]. This feature can be understood in various ways. The most central one is this: the fact that the Shannon measure of information contains the logarithm is related to the postulate that the information gained in a series of observations of different properties must be independent of the specific sequence in which the properties are read out. Clearly, such a requirement is no longer valid in quantum mechanics, unless the properties are commuting, which is in general an exception. So what is needed is a measure of information which accounts for complementarity and describes the total information obtainable in a complete set of quantum experiments. We have suggested elsewhere [20] that the most appropriate measure of information is

$$I = \sum_i p_i p_i \ , \tag{3.4}$$

which may be viewed as the sum of the individual probabilities weighed by these probabilities themselves. The total information content I_{total} of a quantum system is then obtained as a sum of individual measures of information I_j [of the type given in (3.4)] over a complete set of maximally mutually complementary observables (indexed by j)

$$I_{\text{total}} = \sum_j I_j = \sum_j \sum_i p_{ij} p_{ij} \ . \tag{3.5}$$

Here p_{ij} denotes the probability of observing the ith outcome of the jth observable.

The notion of mutually complementary observables may need to be explained further. A famous example is given by the three spin components of a spin-1/2 particle taken along three directions orthogonal in space (not to be confused with orthogonal quantum states). From an operational point of view, two variables A and B are maximally mutually complementary if the knowledge of one completely precludes any knowledge of the other. In the case of spin, if A represents the spin along the z-direction, then B might represent the spin along any direction orthogonal to z in space. It is a well known feature that if the spin along z is well defined, the spin along these other directions is maximally undefined. To come back to our example, the sum $\sum_j$ in (3.5) in the case of a spin-1/2 particle has to be taken along any three spatially orthogonal directions, i.e., $j = x, y, z$.

It has not escaped our attention that (3.5) can be put onto a nicely visualizable foundation if one defines an information space spanned by mutually complementary observables [26]. Then I_{total} just represents the square of the length of a vector in that information space when the square of the length of individual components is just given by I_j.

If, as we have suggested above, quantum physics is about information, then we have to ask ourselves what we mean by a quantum system. It is then imperative to avoid assigning any variant of naive classical objectivity to quantum states [21]. Rather it is then natural to assume that the quantum system is just the notion to which the probabilities in (3.4) and (3.5) refer, and no more. The notion of an independently existing reality thus becomes void.

We might therefore ask how much information a quantum system might carry, stressing again that by 'carry' we just refer to the total amount of information and not to the objective existence of any subject actually carrying the information.

It is obvious that a large system, being our mental representative of the information characterizing it, carries a lot of information, i.e., a great many bits. Then how does that amount of information scale with the size of the object? It is very suggestive to assume that the smaller a system, the less information it carries. One may even consider the amount of information carried by a system as defining its size. Basically, we postulate that:

1. the amount of information carried by any system is finite,
2. the amount of information is lesser the smaller the system.

These assumptions may be supported by referring to Feynman [22]: "It always bothers me that, according to the laws as we understand them today, it takes a computing machine an infinite number of logical operations to figure out what goes on in no matter how tiny a region of space, and no matter how tiny a region of time. How can all that be going on in that tiny space? Why should it take an infinite amount of logic to figure out what one tiny piece of space/time is going to do?" Evidently, Feynman's problem is solved if the 'tiny piece of spacetime' only contains a finite amount of information, and the less, the smaller the piece is.

We arrive at a natural limit when a system only represents one bit of information. Once that is achieved, the system can only represent the yes/no answer to one question. If the system is asked another question, the answer by necessity has to be random. Thus, randomness is a fundamental feature of our world [23, 24]. This, we suggest, also provides a natural foundation for complementarity. Consider, for example, a simple two-path interferometer as shown in Fig. 3.7.

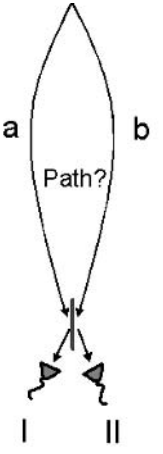

Fig. 3.7. Two-path interferometer. The source emits coherent waves of which two beams are selected and incident on the beam-splitter. Each of the two beams has the same amplitude for being transmitted or reflected at the beam-splitter, and the outgoing beams are thus coherent superpositions of the incoming ones

As is well known, in such an interferometer, we can prepare the state such that either the path $|a\rangle$ or the path $|b\rangle$ is taken by a particle. In that case, the trajectory after the semi-reflecting beam-splitter is completely random, or, in other words, detectors I or II will each register the particle with the same probability of 50%. On the other hand, we can prepare the state in a coherent

superposition of $|a\rangle$ and $|b\rangle$ in such a way that, by adjusting the relative phase, the particle ends up in detector I with certainty. In other words, in the outgoing beam leading to detector I, constructive interference happens and destructive interference in the outgoing beam II. The observation of the path the particle takes inside the interferometer and the observation of the interference are examples of two maximally mutually complementary observations.

This behaviour can be understood very simply on the basis of our most elementary quantum system carrying just one bit of information. It is then up to the experimentalist to decide whether she wants to prepare the system in such a way that the single bit of information is used to completely determine the path, a binary variable, in which case no information is left to determine the fate of the particle after the beam-splitter. Then the outcome, i.e., which detector, I or II, fires, must be completely random. Alternatively, the experimentalist can prepare the system such that the single bit of information defines which detector fires, i.e., the interference, in which case the path is completely undefined. Evidently, intermediate cases are possible, where both path information and interference are each partly defined.

We note that we are thus led to a natural explanation of quantum complementarity. Our measure of information defined in (3.5) also provides a perfect measure for the intermediate cases, where the path is partially defined and also interference is obtained only with partial visibility, so that their information contents sum to a total of one bit [25].

Concluding this chapter, we note that using our approach, we were able to explain some other important features of quantum mechanics, most notably Malus' law [26], which describes the cosine dependence of the probability upon the angle between the measurement direction and the direction along which the spin is well defined. We were also able to obtain a natural understanding of entanglement [27]. For example, if one considers entanglement of two spin-1/2 particles, one has two elementary systems in our sense, and thus two available bits of information. These two bits can be used to encode properties of the individual particles themselves, which is basically classical coding. On the other hand, the two bits can be completely used up to fully define only joint information, that is, information about how possible measurement results on the particles relate to each other. If done in this way, one automatically obtains the four Bell states. Then one obtains a natural basis for Schrödinger's definition of entanglement [28]. Finally, we note that, using our approach, we were able to derive the Liouville equation describing the quantum evolution in time of a two-state system [26].

Clearly, a number of important questions remain open. Of these, we mention here two. The first refers to continuous variables. The problem there is that with continuous variables, one has in principle an infinite number of complementary observables. One might tackle this question by generalizing the definition of (3.4) to infinite sets. This, while mathematically possible,

leads to conceptually difficult situations. The conceptual problem is in our view related to the fact that we wish to define all notions on operationally verifiable bases or foundations, that is, on foundations which can be verified directly in experiment. It is obvious that an infinite number of complementary observables can never be realized in experiment. In our opinion, it is therefore suggestive that the concept of an infinite number of complementary observables and therefore, indirectly, the assumption of continuous variables, are just mathematical constructions which might not have a place in a final formulation of quantum mechanics.

This leads to the second question, namely, how to derive the Schrödinger equation. If the assumption just expressed is correct, namely that continuous variables are devoid of operational and therefore physical meaning in quantum mechanics, there is no need to express the Schrödinger equation based on continuous variables in our new language. Indeed, one should then refer to situations where one always has only a finite number of complementary observables. In our opinion such a point of view is experimentally well founded, as any experiment will always lead to only a finite number of bits and a finite number of the experimental results on the basis of which only a finite number of observables can be operationally defined.

It has not escaped our attention that our way of reasoning also leads to new possibilities for understanding why we have quantum physics, i.e., for answering Wheeler's famous question: Why the quantum? Identifying systems with the information they carry, we note that information is necessarily quantized. One can have one proposition, two propositions, three propositions, etc., but obviously the concept of, say, $\sqrt{2}$ propositions is devoid of any meaning. Therefore, since information is quantized that way, our description of information, which is quantum mechanics, also has to be quantized.

Acknowledgments

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4 Quantum Theory Looks at Time Travel

Daniel M. Greenberger and Karl Svozil

Classically, time travel is inconsistent with free will. If one could visit the past, then one could change the past, and this would lead to an alternative present. So there is a paradox here, which is best illustrated by the famous scenario of a person going back in time to shoot his father before his father has met his mother, and thus negating the possibility of his having ever been born. It is for reasons like this that time travel has been considered impossible in principle [1].

Of course, one can get around this problem if one considers the universe to be totally deterministic, and free will to be merely an illusion. Then the possibility of changing the past (or the future, for that matter) no longer exists. Since we prefer to think that the writing of this paper was not preordained at the time of the big bang, we shall reject this solution on psychological grounds, if not logical ones, and ask whether the paradoxes of classical physics can be gotten around, quantum mechanically.

Most attempts to go beyond the confines of classical theory in order to study time travel have been in the framework of relativity theory, making use of the freedom to warp the topological properties of spacetime. We shall not comment on these here, except to note that they are not incompatible with what we shall be saying, and might conceivably be combined with it.

It seems to us that time travel is very much in the spirit of quantum mechanics, and in fact, it seems quite arbitrary and outside the spirit of the subject to forbid it [2]. For example, if one studies the propagation of a physical system from time t_1 to a later time t_2, one writes

$$\psi(t_2) = U(t_2, t_1)\psi(t_1) \ , \quad t_2 > t_1 \ , \tag{4.1}$$

where U is some unitary operator describing the dynamical unfolding of the system. To calculate U, one sums over all possible paths leading from the initial state to the final state, but restricting these paths to the forward direction of time.

Furthermore, it is well known that when one makes measurements in quantum theory, one's simple sense of causality is violated, and so a classical sense of causality is a rather poor guide as to what should or should not be allowed quantum mechanically. And this restriction would seem to violate the spirit of the entire enterprise. Specifically, why should there not be some

form of feedback into the past in determining what will happen in the future (see Fig. 4.1)?

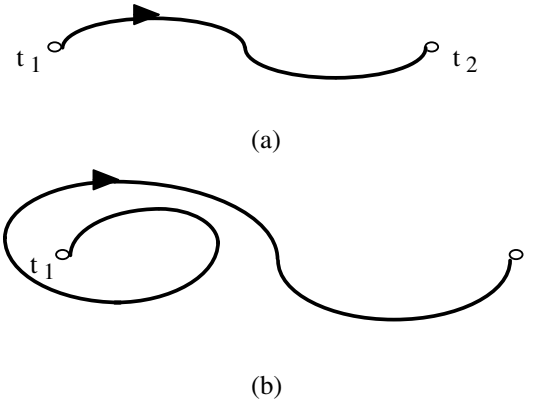

Fig. 4.1. In the path integral one can take all paths (**a**) that go forward in time, but one excludes all paths (**b**) that go backward in time

In order to incorporate some form of feedback into the scheme, a simple feedback mechanism such as that used in electronic circuits would be impossible, because in such a scheme, a simple feedback loop, such as that of Fig. 4.2 is used, and in such a loop, one has two circuit paths feeding into one, and quantum mechanically this would violate unitarity, because it could not be uniquely reversed. However, quantum mechanically, there is another way to introduce feedback, and that is through the introduction of beam splitters, which are unitary.

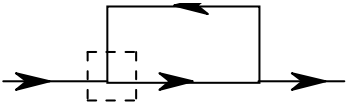

Fig. 4.2. In a classical feedback circuit, one inserts a loop that goes from a later time to an earlier time. The loop then has two entry ports and only one exit port, so that one cannot uniquely reverse it, and if tried quantum mechanically, it would violate unitarity

4.1 Model of a Feedback System in Time

The model that we introduce is one which has two beam splitters, which allows us to generalize the classical scheme of Fig. 4.2, and at the same time

to present a unitary scheme allowing the particle to sample earlier times. This should not be confused with the operation of time reversal, which is an anti-unitary operation. The scheme is shown in Fig. 4.3.

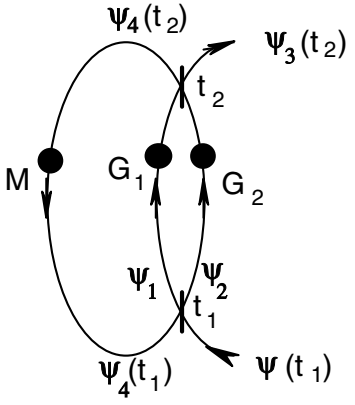

Fig. 4.3. A quantum time evolution scheme with feedback. With no feedback, $\psi(t_1)$ would evolve through G_1 into $\psi_3(t_2)$. There is another evolution channel G_2 and a feedback channel M that alter the output at time t_2

In this scheme, if there were no feedback, then the standard unitary time development would have $\psi(t_1)$ evolving into $\psi_3(t_2)$,

$$\psi_3(t_2) = G_1\psi(t_1)\ . \tag{4.2}$$

Here, the operator M generates the effects of the feedback in time. These 'beam splitters' are figurative, and their role is merely to couple the two incoming channels to two outgoing channels. The operator G_1 represents the ordinary time development in the absence of time feedback. The operator G_2 represents an alternate possible time evolution that can take place and compete with G_1 because there is feedback. We want to find $\psi_3(t_2) = f\big(\psi(t_1)\big)$ in the presence of the feedback in time that is generated by the operator M.

At the beam splitters, which are shown in more detail in Fig. 4.4, the forward amplitude is α, while the reflected amplitude is $\mathrm{i}\beta$. One needs the factor of i because the two amplitudes must differ by 90° in order to preserve unitarity. Normally, we expect that $\alpha \gg \beta$, and in the limit $\alpha = 1$, we should get the situation represented by (4.2).

The beam splitters perform the unitary transformation

$$|a\rangle = \alpha|d\rangle + \mathrm{i}\beta|c\rangle\ , \quad |b\rangle = \alpha|c\rangle + \mathrm{i}\beta|d\rangle\ , \quad \alpha^2 + \beta^2 = 1\ . \tag{4.3}$$

Here we assume for simplicity that α and β are real. We can invert this to obtain

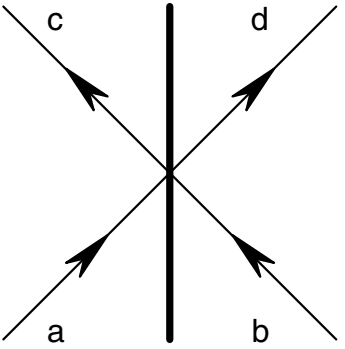

Fig. 4.4. The beam splitter transmits with an amplitude α and reflects with an amplitude $\mathrm{i}\beta$. The factor of i preserves unitarity

$$|d\rangle = \alpha|a\rangle - \mathrm{i}\beta|b\rangle \ , \quad |c\rangle = \alpha|b\rangle - \mathrm{i}\beta|a\rangle \ . \tag{4.4}$$

The overall governing equations can be read directly from Fig. 4.3. At time t_2 the second beam splitter determines $\psi_3(t_2)$ and $\psi_4(t_2)$. We have

$$\psi_3(t_2) \equiv \psi_3' = \left[\alpha\psi_1(t_2) - \mathrm{i}\beta\psi_2(t_2)\right] = \alpha\psi_1' - \mathrm{i}\beta\psi_2' \ , \tag{4.5}$$

where the prime indicates the time t_2 in the argument, and no prime indicates the time t_1. The wave functions ψ_1 and ψ_2 are determined at time t_2 by

$$\psi_1(t_2) = \psi_1' = G_1\psi_1(t_1) = G_1\psi_1 \ , \tag{4.6}$$

$$\psi_2' = G_2\psi_2 \ . \tag{4.7}$$

So that from (4.5),

$$\psi_3' = \alpha G_1\psi_1 - \mathrm{i}\beta G_2\psi_2 \ , \tag{4.8}$$

and equivalently

$$\psi_4' = \alpha G_2\psi_2 - \mathrm{i}\beta G_1\psi_1 \ . \tag{4.9}$$

The propagator M is what produces the feedback in time, propagating from t_2 back to t_1, so that $\psi_4(t_1) = M\psi_4(t_2)$, or

$$\psi_4 = M\psi_4' \ . \tag{4.10}$$

At the t_1 beam splitter,

$$\psi_1 = \alpha\psi - \mathrm{i}\beta\psi_4 \ , \tag{4.11}$$

$$\psi_2 = \alpha\psi_4 - \mathrm{i}\beta\psi \ . \tag{4.12}$$

4.2 The Solution

First, we want to eliminate the ψ_4 in (4.11) and (4.12), to get equations for ψ_1 and ψ_2. Then from (4.8) we can obtain ψ_3'. From (4.9) and (4.10),

$$\psi_4 = M\psi_4' = \alpha M G_2 \psi_2 - \mathrm{i}\beta M G_1 \psi_1 \ . \tag{4.13}$$

We plug this into (4.11) and (4.12),

$$\psi_1 = \alpha\psi - \mathrm{i}\beta(\alpha M G_2 \psi_2 - \mathrm{i}\beta M G_1 \psi_1) \ , \tag{4.14}$$
$$\psi_2 = \alpha(\alpha M G_2 \psi_2 - \mathrm{i}\beta M G_1 \psi_1) - \mathrm{i}\beta\psi \ . \tag{4.15}$$

We can rewrite these as

$$\psi_1 = (1+\beta^2 M G_1)^{-1}(-\mathrm{i}\alpha\beta M G_2)\psi_2 + \alpha(1+\beta^2 M G_1)^{-1}\psi \ , \tag{4.16}$$
$$\psi_2 = (1-\alpha^2 M G_2)^{-1}(-\mathrm{i}\alpha\beta M G_1)\psi_1 - \mathrm{i}\beta(1-\alpha^2 M G_2)^{-1}\psi \ . \tag{4.17}$$

These are two simultaneous equations that we must solve to find ψ_1 and ψ_2 as functions of ψ. To solve for ψ_1, substitute (4.17) into (4.16),

$$\psi_1 = (1+\beta^2 M G_1)^{-1}(-\mathrm{i}\alpha\beta M G_2)\Big[(1-\alpha^2 M G_2)^{-1}(-\mathrm{i}\alpha\beta M G_1)\psi_1 \\ -\mathrm{i}\beta(1-\alpha^2 M G_2)^{-1}\psi\Big] + \alpha(1+\beta^2 M G_1)^{-1}\psi \ . \tag{4.18}$$

This can be rewritten as

$$\left[1+\alpha^2\beta^2(1+\beta^2 M G_1)^{-1}(M G_2)(1-\alpha^2 M G_2)^{-1}(M G_1)\right]\psi_1 \\ = (1+\beta^2 M G_1)^{-1}\left[-\alpha\beta^2 M G_2(1-\alpha^2 M G_2)^{-1}+\alpha\right]\psi \ . \tag{4.19}$$

If we write this as

$$[X]\psi_1 = Y^{-1}[Z]\psi \ , \tag{4.20}$$

then we can simplify the equation as follows:

$$\begin{aligned} YX &= 1+\beta^2 M G_1 + \alpha^2\beta^2 M G_2(1-\alpha^2 M G_2)^{-1} M G_1 \\ &= 1+\beta^2\left[1+(1-\alpha^2 M G_2)^{-1}\alpha^2 M G_2\right] M G_1 \\ &= 1+\beta^2(1-\alpha^2 M G_2)^{-1} M G_1 \ , \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} Z &= \alpha(1-\alpha^2 M G_2)^{-1}(1-\alpha^2 M G_2 - \beta^2 M G_2) \\ &= \alpha(1-\alpha^2 M G_2)^{-1}(1-M G_2) \ . \end{aligned} \tag{4.22}$$

Thus,

$$\psi_1 = \alpha \left[1 + \beta^2 (1 - \alpha^2 M G_2)^{-1} M G_1\right]^{-1} (1 - \alpha^2 M G_2)^{-1} (1 - M G_2)\psi \; . \tag{4.23}$$

Then, using the identity $A^{-1}B^{-1} = (BA)^{-1}$, we finally get

$$\psi_1 = \alpha (1 - \alpha^2 M G_2 + \beta^2 M G_1)^{-1} (1 - M G_2)\psi \; . \tag{4.24}$$

We can solve for ψ_2 similarly, by substituting (4.16) into (4.17),

$$\psi_2 = -\mathrm{i}\beta (1 - \alpha^2 M G_2 + \beta^2 M G_1)^{-1} (1 + M G_1)\psi \; . \tag{4.25}$$

Notice that in the denominator term in both (4.24) and (4.25), α and β have reversed the role of the operators they apply to. We can finally use (4.8) to solve for $\psi_3' = \psi_3(t_2)$,

$$\psi_3(t_2) = \left[\alpha^2 G_1 D (1 - M G_2) - \beta^2 G_2 D (1 + M G_1)\right] \psi(t_1) \; , \tag{4.26}$$

where $D = (1 + \beta^2 M G_1 - \alpha^2 M G_2)^{-1}$.

4.3 Some Important Special Cases

The Case $\alpha = 1$, $\beta = 0$. This is the case where there is no feedback. Here

$$\psi_3' = G_1 (1 - M G_2)^{-1} (1 - M G_2)\psi = G_1 \psi \; . \tag{4.27}$$

The Case $\beta = 1$, $\alpha = 0$. This is the case where there is only feedback. Here

$$\psi_3' = -G_2 (1 + M G_1)^{-1} (1 + M G_1)\psi = -G_2 \psi \; . \tag{4.28}$$

The Case $G_1 = G_2 \equiv G$.

$$\psi_3' = G \left[1 + (\beta^2 - \alpha^2) M G\right]^{-1} (\alpha^2 - \beta^2 - M G)\psi \; . \tag{4.29}$$

If we also have $\alpha^2 = \beta^2 = 1/2$, then

$$\psi_3' = -G M G \psi \; . \tag{4.30}$$

The Case $\beta \ll 1$. This is expected to be the usual case. Then the answer only depends on $\beta^2 = \gamma$. Also, $\alpha^2 = 1 - \beta^2 = 1 - \gamma$. Then to lowest order in γ, the denominator D in (4.26) becomes

$$\begin{aligned} D &= \left[1 + \gamma M G_1 - (1 - \gamma) M G_2\right]^{-1} \\ &= (1 - M G_2)^{-1} - \gamma (1 - M G_2)^{-1} (M G_1 + M G_2)(1 - M G_2)^{-1} \; , \end{aligned} \tag{4.31}$$

so that

$$\begin{aligned} \psi_3' &= \Big\{ (1 - \gamma) G_1 \left[1 - \gamma (1 - M G_2)^{-1} (M G_1 + M G_2)\right] \\ &\qquad - \gamma G_2 (1 - M G_2)^{-1} (1 + M G_1) \Big\} \psi \\ &= G_1 \psi - \gamma (G_1 + G_2)(1 - M G_2)(1 + M G_1)\psi \; . \end{aligned} \tag{4.32}$$

4.4 The Classical Paradox of Shooting your Father

The most interesting case is the one that corresponds to the classical paradox where you shoot your father before he has met your mother, so that you can never be born. This case has a rather fascinating quantum-mechanical resolution. This is the case $G_1 = 0$, where there is a perfect absorber in the beam so that the system without any feedback would never get to evolve to time t_2. But quantum mechanically, we assume that there is another path along G_2, the one where you do not shoot your father, that has a probability β without feedback. In quantum theory we deal with probabilities, and as long as there is any chance that you may not meet your father, we must take this into account.

The solution in this case is

$$\psi_3' = -\beta^2 G_2 (1 - \alpha^2 M G_2)^{-1} \psi \ . \tag{4.33}$$

We assume for simplicity that G_2 is just the standard time evolution operator

$$G_2 = \mathrm{e}^{-\mathrm{i}E(t_2 - t_1)/h} \ . \tag{4.34}$$

and M is just the simplest backwards in time evolution operator

$$M = \mathrm{e}^{-\mathrm{i}E(t_1 - t_2)/h + \mathrm{i}\phi} \ , \tag{4.35}$$

where we have also allowed for an extra phase shift. Then

$$\psi_3' = -\beta^2 \mathrm{e}^{-\mathrm{i}E(t_2 - t_1)/h} \left(1 - \alpha^2 \mathrm{e}^{\mathrm{i}\phi}\right)^{-1} \psi \ , \tag{4.36}$$

$$|\psi_3'|^2 = \frac{\beta^4}{(1 - \alpha^2 \mathrm{e}^{\mathrm{i}\phi})\,(1 - \alpha^2 \mathrm{e}^{-\mathrm{i}\phi})} |\psi|^2 = \frac{1}{1 + 4(\alpha^2/\beta^4)\sin^2(\phi/2)} |\psi|^2 \ . \tag{4.37}$$

Note that for $\phi = 0$, $\psi_3' = -\mathrm{e}^{-\mathrm{i}E\Delta t/h}\psi$, for any value of β. That means that no matter how small the probability of your ever having reached here in the first place, the fact that you are here, which can only happen because $\alpha \neq 1$, guarantees that even though you are certain to have shot your father if you had met him ($G_1 = 0$), nonetheless you will not have met him! You will have taken the other path, with 100% certainty. Obviously, this must be the case, if you are to be here at all.

How can we understand this result? In our model, with $\phi = 0$, we have $G_1 = 0$, and $MG_2 = 1$. Also, we will assume that $\beta \ll 1$, even though this is not necessary. The various amplitudes are

$$|\psi_1| = 0 \ , \quad |\psi_2/\psi| = 1/\beta \ , \quad |\psi_4/\psi| = \alpha/\beta \ , \quad |\psi_3'/\psi| = 1 \ . \tag{4.38}$$

So we see that the two paths of the beam splitter at t_1 leading to the path ψ_1 cancel out. But of the original beam ψ, α passes through to ψ_1, while

of the beam ψ_4, only the fraction β leaks through to ψ_1. So the beam ψ_4 must have a very large amplitude, which it does, as we can see from (4.38), so that the two contributions can cancel at ψ_1. In fact ψ_4 has a much larger amplitude than the original beam! Similarly, in order to have $|\psi_3'| = |\psi|$, then ψ_2 must have a very large amplitude. Thus we see that there is a large current flowing around the system, between ψ_2 and ψ_4. But does this not violate unitarity? The answer is that if they were both running forward in time, it would. But one of these currents is running forward in time, while the other runs backward in time, and so they do not in this case violate unitarity. This is how our solution is possible.

4.5 Conclusion

According to our model, if you could travel into the past quantum mechanically, you would only see those alternatives consistent with the world you left behind you. In other words, while you are aware of the past, you cannot change it. No matter how unlikely the events are that could have led to your present circumstances, once they have actually occurred, they cannot be changed. Your trip would set up resonances that are consistent with the future that has already unfolded.

This also has enormous consequences on the paradoxes of free will. It shows that it is perfectly logical to assume that one has many choices and that one is free to take any one of them. Until a choice is taken, the future is not determined. However, once a choice is taken, and it leads to a particular future, it was inevitable. It could not have been otherwise. The boundary condition that the future events happened as they already have, guarantees that they must have been prepared for in the past. So, looking backwards, the world is deterministic. However, looking forwards, the future is probabilistic. This completely explains the classical paradox. In fact, it serves as a kind of indirect evidence that such feedback must actually take place in nature, in the sense that without it, a paradox exists, while with it, the paradox is resolved. (Of course, there is an equally likely explanation, namely that going backward in time is impossible. This also solves the paradox by avoiding it.)

The model also has consequences concerning the many-worlds interpretation of quantum theory. The world may appear to keep splitting so far as the future is concerned. However, once a measurement is made, only those histories consistent with that measurement are possible. In other words, with time travel, other alternative worlds do not exist, as once a measurement has been made confirming the world we live in, the other worlds would be impossible to reach from the original one. This explanation makes the von Neumann state reduction hypothesis much more reasonable, and in fact acts as a sort of justification of it.

Another interesting point comes from examining (4.37). For small angles ϕ, we see that

$$|\psi_3'|^2 = \frac{1}{1 + 4(\alpha^2/\beta^4)\sin^2(\phi/2)}|\psi|^2 \longrightarrow \frac{1}{1 + \alpha^2\phi^2/\beta^4}|\psi|^2 \,, \tag{4.39}$$

so that the above result is strongly resonant, with a Lorentzian shape, and a width $\Delta\phi \sim \beta^2$, since $\alpha \sim 1$. Thus less 'deterministic' and fuzzier time travelling might be possible, a possibility we have not yet explored. Neither have we explored the possibility that feedback should be possible into the future as well as the past. Of course in this case, it ought to be called 'feedforward', rather than feedback.

References

1. There are many books about the nature of time. The two main paradoxes are the question of reversing the direction of time so that time travel is possible, and the issue of the 'arrow of time', namely, why time flows in one direction. A book of interesting essays on both questions is *The Nature of Time*, ed. by R. Flood and M. Lockwood, Basil Blackwell, Cambridge, Mass. (1986). Some other interesting references are: *Time's Arrow and Archimedes' Point*, by H. Price, Oxford University Press, New York (1996); *The Physical Basis of the Direction of Time*, by H.D. Zeh, Springer-Verlag, Berlin (1999); and *Time's Arrows and Quantum Measurement*, by L.S. Schulman, Cambridge University Press, Cambridge (1997)
2. This paper is an expanded version of an earlier paper on the subject, D.M. Greenberger and K. Svozil, in: *Between Chance and Choice*, ed. by H. Atmanspacher and R. Bishop, Imprint Academic, Thorverton (England) (2002)

5 What Connects Different Interpretations of Quantum Mechanics?

James B. Hartle

In this paper, we investigate the idea that different interpretations of quantum mechanics can be seen as restrictions of the consistent (or decoherent) histories quantum mechanics of closed systems to particular classes of histories, together with the approximations and descriptions of these histories that the restrictions permit.

5.1 Introduction

The authors of this book have jointly identified as many as thirteen different interpretations of QM [1]. The very length of this list invites the questions:

- What are the relationships between these interpretations?
- To what uses may they be put?
- Is it possible to settle objectively on one?

This brief article offers some personal reflections on these questions.

The defining thread connecting interpretations of quantum theory is their agreement on the probabilities for the outcomes of measurements, at least to an excellent approximation. Some formulations may provide probabilities for further kinds of alternatives such as the position of the Moon when it is not receiving attention from observers, or the values of density fluctuations in the very early universe when there were no observers around. However, a formulation that does not reproduce the standard textbook answers for the probabilities of measurements is not an interpretation of quantum mechanics. Rather, it is a different theory. Such alternatives to quantum theory are of great interest but not the subject of this essay.

The idea explored here is that a number of different interpretations of quantum mechanics can be connected through the consistent (or decoherent) quantum mechanics of a closed system. Specifically, a number of interpretations can be seen as restrictions of consistent histories quantum theory to particular kinds of sets of alternative histories[1] together with the approximations and special descriptions of the sets that these restrictions permit. This essay examines three cases where this connection can be made and gives brief discussions of the utility of the restrictions involved.

[1] 'Frameworks' in the terminology of Griffiths.

5.2 Quantum Mechanics of Closed Systems

We begin with a very brief review of the quantum mechanics of a closed system, most generally the universe as a whole.[2] To simplify the discussion we neglect quantum gravity and assume a fixed background spacetime geometry. The familiar apparatus of Hilbert space, states, and operators may then be employed to formulate the quantum mechanics of the closed system.[3] As a simple model, we can think of a large, isolated box of N non-relativistic particles. Dynamics can be specified in terms of particle positions $\boldsymbol{x}_i$ and momenta $\boldsymbol{p}_i$ by a Hamiltonian

$$H = \sum_{i=1}^{N} \frac{\boldsymbol{p}_i^2}{2m_i} + V(\boldsymbol{x}_i) \; . \tag{5.1}$$

Both observers and observed, if any, are contained inside. This is evidently not the most general description of a closed system but it will suffice to illustrate some of the connections between interpretations that we describe later.

We take the closed system to be described by a quantum state $|\Psi\rangle$. The most general objective of quantum theory is the prediction from H and $|\Psi\rangle$ of the probabilities of the individual members of a set of coarse-grained alternative histories of the closed system. A history is described by giving a sequence of alternatives $(\alpha_1, \ldots, \alpha_n)$ at a series of times $t_1, \ldots, t_n$. Alternatives at a moment of time t_k are represented by an exhaustive set of orthogonal, Heisenberg picture, projection operators $\{P^k_{\alpha_k}(t_k)\}$, $\alpha_k = 1, 2, \ldots$, and a history of alternatives is represented by the corresponding chain of projections called a class operator:

$$C_\alpha = P^k_{\alpha_n}(t_n) \ldots P^1_{\alpha_1}(t_1) \; . \tag{5.2}$$

Note that, on the left-hand side of (5.2), we have abbreviated the whole chain $(\alpha_1, \ldots, \alpha_n)$ by a single index α. For example, if we are interested in a history of the Earth moving around the Sun, the P's might be projections onto exclusive ranges of the center of mass position of the Earth at a sequence of times. This set of histories is coarse-grained because alternatives are not specified at every time but only at some times, because the center of mass position is not specified exactly but only in certain ranges, and because not every variable describing the universe is specified but only the center of mass of the Earth.

The class operators C_α defined in (5.2) permit the construction of branch state vectors,

[2] For details, see the classic expositions in [2–4].

[3] For the generalizations that may be otherwise required, see [5]. We view the quantum mechanics of closed systems as an extension and completion of the Everett formulation and therefore do not count that as a separate interpretation.

$$|\Psi_\alpha\rangle = C_\alpha|\Psi\rangle \ , \tag{5.3}$$

for each history in the coarse-grained set. A set of histories decoheres when there is negligible mutual interference between all the branch state vectors:

$$\langle\Psi_\alpha|\Psi_{\alpha'}\rangle \approx 0 \ , \quad \alpha \neq \alpha' \ . \tag{5.4}$$

The joint probability p_α for all the events in a history α is

$$p_\alpha = \||\Psi_\alpha\rangle\|^2 = \|C_\alpha|\Psi\rangle\|^2 \ , \tag{5.5}$$

when the set of histories decoheres. Decoherence ensures the validity of the probability sum rules which are among the defining properties of probability.

The above discussion is brief, certainly oversimplified in some respects, but sufficient we hope for understanding the remarks which follow. The key point for the ensuing discussion is the following. Decoherent histories quantum mechanics predicts probabilities for many different sets of alternative histories which are complementary in the following sense: each set is part of a complete quantum description of the system, but there is no fine-grained decoherent set of histories of which all the decoherent sets are coarse-grainings. A set of histories coarse-grained by the Earth's center of mass momentum is an example of a set which (if decoherent) would be complementary to the set coarse-grained by the Earth's center of mass position.

Given H and $|\Psi\rangle$, it is in principle possible to calculate all decoherent sets. Among these is the quasiclassical realm of everyday experience, coarse-grained by the variables of classical physics, and exhibiting classical patterns of correlation in time, summarized approximately by classical equations of motion. As human observers we focus almost entirely on coarse-grainings of this quasiclassical realm. However, quantum theory does not distinguish the quasiclassical realm from other decoherent sets except by properties such as its classicality.

The picture of quantum reality which emerges from the quantum mechanics of closed systems is very different from the reality of classical physics involving, as it does, many complementary descriptions of the universe that are mutually incompatible. Restricting the allowed sets of histories by some principle[4] typically yields a description of reality that is closer in character to the familiar classical one. We will see this in the cases to be discussed.

5.3 Three Case Studies

This section considers the idea offered in Sect. 5.1 for three different interpretations of quantum theory.

[4] A set selection principle, in the terminology of Dowker and Kent [6].

5.3.1 Copenhagen Quantum Mechanics

The Copenhagen quantum mechanics found in most textbooks is concerned with the probabilities of histories of the outcomes of measurements carried out by observers. The subsystem being observed is described by a Hilbert space $\mathcal{H}_{\rm s}$. Dynamics is specified by a Hamiltonian h acting on $\mathcal{H}_{\rm s}$ when the subsystem is isolated. Initially the subsystem is assumed to be a state $|\psi\rangle$ in $\mathcal{H}_{\rm s}$. The outcomes of a measurement carried out at time t_k can be described by a set of orthogonal, Heisenberg picture, projection operators $\{s^k_{\alpha_k}(t_k)\}$, $\alpha_k = 1, 2, \ldots$, analogous to the P's described in Sect. 5.2. The probabilities for a history of ideal measurements (ones that disturb the subsystem as little as possible) at times $t_1, \ldots, t_n$ are given by the analog of (5.5):

$$p_\alpha = \left\| s^n_{\alpha_n}(t_n) \ldots s^1_{\alpha_1}(t_1)|\psi\rangle \right\|^2 \ . \tag{5.6}$$

Consistency is not an issue for these probabilities. Probabilities for a coarser-grained history need not be the sum of the probabilities of finer-grained histories consistent with it. Finer- and coarser-grained measurements correspond to different interactions of the subsystem with an external apparatus. Sets of histories describing alternative measurements do not have to decohere.

Copenhagen quantum mechanics is an approximation to the quantum mechanics of closed systems that is appropriate for histories of measurement situations when the decoherence of alternatives that register the outcomes of the measurements can be idealized as exact. We sketch only the key features of a demonstration which are essentially the same as many measurement models. For details see, e.g., [7], Sect. II.10.

We consider a closed system with a Hilbert space $\mathcal{H}_{\rm s} \otimes \mathcal{H}_{\rm r}$, where $\mathcal{H}_{\rm s}$ is the Hilbert space of the measured subsystem and $\mathcal{H}_{\rm r}$ is the Hilbert space of the rest of the universe, including any measuring apparatus and observers. We assume an initial state of the form $|\Psi\rangle = |\psi\rangle \otimes |\Phi_{\rm r}\rangle$ and consider a sequence of measurements at a series of times $t_1, \ldots, t_n$. Measured alternatives of the subsystem are described by projection operators whose Schrödinger picture representatives have the form $S^k_{\alpha_k} = s^k_{\alpha_k} \otimes I_{\rm r}$. In a typical measurement situation, an alternative such as $S^k_{\alpha_k}$ becomes correlated with an alternative of the apparatus and in particular with persistent records of the measurements. The orthogonality and persistence of these records guarantees the decoherence of the histories of measured outcomes. If the usual assumption is made that the measurement interaction disturbs the subsystem as little as possible (ideal measurement), then

$$p_\alpha = \left\| S^n_{\alpha_n}(t) \ldots S^1_{\alpha_1}(t_1)|\Psi\rangle \right\|^2_{\mathcal{H}} \approx \left\| s^n_{\alpha_n}(t_n) \ldots s^1_{\alpha_1}(t_1)|\psi\rangle \right\|^2_{\mathcal{H}_{\rm r}} \ . \tag{5.7}$$

Thus Copenhagen quantum mechanics is recovered as a restriction of, and approximation to, the quantum mechanics of closed systems. The second equality in (5.7) is not exact but true to an excellent approximation in realistic

measurement situations – typically far beyond the accuracy with which the probabilities can be checked or the physical situation modeled.

The utility of the approximate quantum mechanics of measured subsystems is evident. It is a truly excellent approximation for every laboratory experiment which has tested the principles of quantum theory. Further, the calculations of the approximate Copenhagen probabilities utilizing just the Hilbert space of the measured subsystem will generally be vastly simpler than in the Hilbert space of the universe. These advantages, however, should not obscure the utility of embedding the Copenhagen quantum mechanics in the more general quantum mechanics of closed systems for understanding measurements (as above) and calculating just how good an approximation it is.

5.3.2 Bohm Theory

To summarize the features of Bohm theory [8] that are relevant to the present discussion, it is convenient to restrict attention to the closed system consisting of N, non-relativistic particles in a box discussed in Sect. 5.2. An initial wave function $\Psi(\boldsymbol{x}_1, \ldots \boldsymbol{x}_N, 0)$ is given. The particles in the box move on trajectories $\boldsymbol{x}_i(t)$ that obey two deterministic equations. The first is the Schrödinger equation for Ψ:

$$\mathrm{i}\hbar \frac{\partial \Psi}{\partial t} = H\Psi \; . \tag{5.8}$$

Then, writing $\Psi = R\exp(\mathrm{i}S)$ with R and S real, the second equation is the deterministic equation for the $\boldsymbol{x}_i(t)$

$$m_i \frac{\mathrm{d}\boldsymbol{x}_i}{\mathrm{d}t} = \boldsymbol{\nabla}_{\boldsymbol{x}_i} S(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \; . \tag{5.9}$$

The initial wave function is the initial condition for (5.8). The theory becomes a statistical theory with the assumption that the initial values of the $\boldsymbol{x}_i$ are distributed according to the probability density on configuration space

$$\wp\,(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N, 0) = |\Psi\,(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N, 0)|^2 \qquad \text{at the initial time } 0 \; . \tag{5.10}$$

Once this initial probability distribution is fixed, the probability of any later alternatives is fixed by the deterministic equation (5.9).

A coarse-grained Bohmian history $\alpha \equiv (\alpha_n, \ldots, \alpha_1)$ is defined by a sequence of ranges $\{\Delta^k_{\alpha_k}\}$ of the $\boldsymbol{x}_i$ at a series of times $t_1, \ldots t_n$ and consists of the set of Bohmian trajectories $\boldsymbol{x}_i(t)$ that cross those ranges at the specified times.

The predictions of Bohm theory and the quantum mechanics of closed systems can be compared for sets of alternative histories coarse-grained by ranges of the position $\boldsymbol{x}_i$ at different times as above. Generally different probabilities are predicted for the same set of histories [9]. This difference arises

as follows: Bohm histories are deterministic. That means that the probability that the particles traverse a series of regions of configuration space at a sequence of times is the same as the probability of the initial values of $\boldsymbol{x}_i$ that evolve to those trajectories under the equations of motion (5.8) and (5.9). The probability of a Bohm trajectory can therefore be represented as

$$p_\alpha^{(\mathrm{BM})} = \|B_\alpha|\Psi\rangle\|^2 \ , \tag{5.11}$$

where B_α is a projection onto the appropriate initial conditions.

The probabilities of the same set of histories would be calculated in decoherent histories quantum theory from [see (5.5)]

$$p_\alpha^{(\mathrm{DH})} = \|C_\alpha|\Psi\rangle\|^2 \ , \tag{5.12}$$

provided the set is decoherent. Here the C_α are chains of projections like (5.2). It is a simple observation is that a chain of projections like (5.2) is not generally a projection and that therefore $p_\alpha^{(\mathrm{BM})}$ will not agree generally with $p_\alpha^{(\mathrm{DH})}$ (see [9] for examples and further discussion).

Another way of seeing the difference is to note that in Bohm theory the wave function always evolves by the Schrödinger equation – unitary evolution. But the action of a chain of projections C_α on the initial state can be described as unitary evolution interrupted by the action of the projections (reduction).

Only in the case of histories with alternatives at a single time are the predictions of Bohm theory and the quantum mechanics of closed systems guaranteed to agree. Then the C_α are projections. But this is an important case because it leads to the conclusion that Bohm theory and the quantum mechanics of closed systems agree on the probabilities of the outcome of measurements.

One characteristic of a measurement situation which seems generally agreed upon is that the results of a measurement are recorded – at least for a time. A history C_α of measurement outcomes is recorded in a set of alternatives $\{R_\alpha\}$ at a time later than the last alternative in C_α if the values of the R_α are correlated with the outcomes of the measurements described by the C_α. The R_α's are projections even if the C_α's are not. Bohm theory and the quantum theory of closed systems will therefore agree on the probabilities of these records.

Bohm theory can therefore be regarded as a restriction of the quantum theory of closed systems to alternatives describing the records of measurements (in the $\boldsymbol{x}$'s) together with the description of these outcomes in terms of deterministic trajectories obeying (5.8) and (5.9). An advantage of Bohm theory (that is, of this restriction) that we believe would be claimed by its proponents is the clear specification of one set of histories (of the $\boldsymbol{x}$'s) as preferred over others. A potential disadvantage is that these histories, although deterministic, may not be classical even in situations where the correlations of

classical physics in time are predicted with high probability by the quantum mechanics of closed systems [10]. Thus, for example, even when a classical past is retrodicted from present records from the quantum mechanics of closed systems, Bohm theory may predict a non-classical one depending on the nature of the initial condition [9].

5.3.3 Sum-Over-Histories

The starting point for a sum-over-histories formulation of quantum mechanics is the specification of one set of fine-grained histories. For the model universe of non-relativistic particles in a box, these are the particle paths $\boldsymbol{x}_i(t)$, $i = 1, \ldots, N$. The allowed coarse-grainings are partitions of this set of fine-grained histories into an exhaustive set of exclusive classes. For example, the paths could be partitioned by how they traverse a set of regions of configuration space $\{\Delta_{\alpha_k}\}$, $\alpha_k = 1, 2, \ldots$, at a sequence of times t_k, $k = 1, \ldots, n$. The class operators C_α are specified by giving their matrix elements as sums over the fine-grained paths in the coarse-grained class labeled by α. Denoting a point in the $3N$-dimensional configuration space by x, this sum is

$$\langle x''|C_\alpha|x'\rangle = \int_\alpha \delta x \, \mathrm{e}^{\mathrm{i}S[x(t)]/\hbar} \, . \tag{5.13}$$

Here, $S[x(t)]$ is the action functional and the sum is over all fine-grained histories in the class labeled by α. For instance, in the partition by sequences of sets of regions at a series of times, a coarse-grained history α is labeled by the regions $(\alpha_1, \ldots, \alpha_n)$ crossed at the sequence of times, and the sum in (5.13) defining the class operator is over paths that cross these regions. The construction of probabilities is then as described in Sect. 5.2.

Sum-over-histories quantum theory is evidently a restriction of the quantum mechanics of closed systems described in Sect. 5.2. All the possible sets of projection operators that might occur in the construction of a set of alternative histories like (5.1) are restricted to projections on ranges of position. The predictions of the restricted sets agree because of identities that express sums-over-histories in terms of operators. For instance,

$$\int_{[x'', \Delta_n, \ldots, \Delta_1, x']} \mathrm{d}x \, \mathrm{e}^{\mathrm{i}S[x(t)]/\hbar} = \left\langle x'' \left| P^n_{\Delta_n}(t_n) \ldots P^1_{\Delta_1}(t_1) \right| x' \right\rangle \, , \tag{5.14}$$

where the sum on the left-hand side is over all paths that start at x' pass through the regions $(\Delta_1, \ldots, \Delta_n)$ at times $t_1, \ldots, t_n$, and end at x' [11].

The sum-over-histories formulation of quantum theory is not usually discussed as a different interpretation of quantum mechanics. But it can be [12, 13] because, like Bohm theory, it specifies a fundamental set of variables. In effect, it posits a set selection principle. To the extent that the quasiclassical realm in which we operate as human observers can be described as a coarse-graining of configuration space [14], no predictive power is lost in

making this restriction. However, the restriction is not so strong as to narrow the range of available sets just to the quasiclassical realm.

There is some loss in convenience with a sum-over-histories formulation because quantities like the momentum of a particle must be described in spacetime terms – by time of flight for example [12]. But there is also potential gain. A sum-over-histories restriction provides a head start in the characterization of classicality and the explanation of its origin (see e.g., [15]). A sum-over-histories formulation of quantum mechanics is the natural framework for investigating generalizations of quantum mechanics that are necessary to describe spacetime alternatives extended over time (e.g., [16]) and those which may be needed for a quantum theory of gravity [5, 17].

5.4 Is There One Interpretation of Quantum Mechanics?

It would be interesting to investigate how many different interpretations of quantum theory can be seen as restrictions of the quantum mechanics of closed systems together with the approximations and particular descriptions of histories that these restrictions permit. That would be at least one way of connecting different interpretations and a common basis for discussing their assumptions, advantages, motivations, and limitations.

It would be equally interesting to identify interpretations of quantum mechanics which cannot be viewed as restrictions of the quantum mechanics of closed systems for some fundamental reason (and not simply because they lack the coherence to decide). Consistent histories quantum mechanics is logically consistent, consistent with experiment as far as is known, consistent with textbook predictions for measurements, and applicable to the most general physical systems. However, it may not be the only theory with these properties. Investigations of interpretations that do not fit within its umbrella framework may lead in different directions.

Can we distinguish between the different interpretations that are restrictions of the quantum mechanics of closed systems? Not by experiment or observations. By assumption, the different interpretations agree on the predictions for measurement to excellent approximations. It seems unlikely to this author that we can settle on one interpretation by argument and discussion. (There is some empirical evidence for this conclusion.) There are too many individually held opinions on the objectives to be met by the restrictions. But neither does there seem to be a compelling need to settle among interpretations that are restrictions of a common quantum mechanics of closed systems.

We may be able to distinguish interpretations by their utility and/or their promise as starting points for generalizations or alternatives to quantum theory. For instance, Copenhagen quantum mechanics is inadequate for cosmology. In cosmology there is no fundamental division of the closed system into two parts, one of which measures the other. Measurements and

observers cannot be fundamental in a theory that seeks to describe the early universe where neither existed. In a quantum world there are generally no variables that behave classically in all circumstances. As another example, sum-over-histories quantum theory may be a productive route to generalizing usual quantum theory to incorporate the dynamical spacetime geometry of quantum gravity [5].

Many years ago, when an instructor at Princeton, I discussed my first effort in understanding quantum mechanics [18] with Eugene Wigner. At the conclusion of the discussion I asked him whether I should publish my results. Wigner explained that there were some subjects – and the interpretation of quantum mechanics was one of these – that one could not learn about by reading books or attending lectures. One just had to work through them by oneself. And usually, if people took the trouble to do this and reached a conclusion, they published a paper. "So", he said, "why shouldn't you?" Maybe that is another reason there are so many interpretations of quantum theory.

Acknowledgments

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6 Is Quantum Mechanics the Whole Truth?

A.J. Leggett

I first briefly review the reasons for speculating that the linear formalism of standard quantum mechanics may break down at some stage between the level of the atom and that of human consciousness. Next, I review the existing experimental evidence on this question, with particular emphasis on the spectacular advances of the last two years in quantum optics and condensed matter physics. Finally, I speculate on possible directions for future experiments in this area.

6.1 Introduction

Most of the authors in this book have taken for granted that at the end of the 21st century the general framework provided by quantum mechanics will still be regarded as universally valid, and have speculated on ways in which this framework may be developed, extended or interpreted. In this contribution I want to take a rather different point of view: to ask whether it is possible and even likely that quantum mechanics may be discovered not to be the whole truth about the physical world, and to enquire what constraints may be put on such a possible failure from existing and future experiments. I have recently written an extended review of this whole subject (Leggett 2002), and will therefore not go into great detail here on the relevant experiments (though I will update the above reference somewhat, since there have been exciting developments over the last year).

6.2 Motivation

Why should we even contemplate the possibility that quantum mechanics (hereafter abbreviated QM) may not be the whole truth about the physical world? At least for me, the main motivation is the existence, and (in my view) insolubility within the standard scheme of QM, of the measurement paradox. To summarize an argument which is given in much more detail in Sect. 2 of (Leggett 2002):

1. At the microscopic level (the level of single electrons, neutrons etc.) a linear superposition of the form $a\psi_1 + b\psi_2$ cannot be reasonably interpreted

as implying that each electron (etc.) of the ensemble occupies either the state described by ψ_1 (with probability $|a|^2$) or that described by ψ_2 (probability $|b|^2$); the evidence that such an interpretation is unviable is provided by the phenomenon of interference between the states ψ_1 and ψ_2, as manifested, e.g., in a standard Young's slits experiment.

2. The assumption that the formalism of QM is of universal validity leads to the generation, under suitable experimental conditions, of Schrödinger's-cat-type states, that is, states of the universe of the form $a\Psi_1 + b\Psi_2$, where Ψ_1 and Ψ_2 describe states of the world which are by some reasonable criterion macroscopically distinct (e.g. 'cat alive' and 'cat dead').
3. The fact that at this (macroscopic) level the evidence, in the form of observable interference between the states Ψ_1 and Ψ_2, against the interpretation rejected at stage (1) has become impossible to obtain in no way justifies us in now embracing this interpretation. (The formalism of QM is a seamless whole, and makes no distinction between microscopic and macroscopic lines, so any interpretation of it should likewise be seamless.)
4. Therefore, it is impossible to interpret the superposition $a\Psi_1 + b\Psi_2$ as saying that in any particular instance (i.e., on any one 'run' of the many experimental runs composing the ensemble so described) the universe is either in the macroscopic state described by Ψ_1 (with probability $|a|^2$) or in that described by Ψ_2 (with probability $|b|^2$). This conclusion appears to conflict with our 'common-sense' perception (?) that at the macroscopic level, at least, the universe is always in a definite state.

Of the various 'resolutions' of the apparent paradox which have appeared in the literature there are three which seem to me to represent in some sense 'pure' cases and which I shall therefore mention explicitly here; most of the others are in some sense blends of these arguments with different weights.

The first is the Everett–Wheeler (relative-state, 'many-worlds') interpretation: this accepts all four stages of the above argument, and while agreeing that the conclusion apparently conflicts with common sense, argues that common sense is wrong: the universe, when described by a Schrödinger's cat wave function, really is not in a definite macroscopic state, and our impression to the contrary is an illusion. I cannot claim to understand this interpretation, since the words 'really is (not)', as used by its advocates, while ostensibly English, convey nothing to me.

The second interpretation, the extreme 'statistical' interpretation as advocated, e.g., by Ballentine (1970), may perhaps be regarded as the view one arrives at by taking the traditional (and not very well defined) Copenhagen interpretation to its logical conclusion: in effect it challenges the above argument already at stage 1, by denying that the very question "did this electron go through slit 1 or slit 2?" has any meaning. According to this view, the only 'reality' is constituted by directly observed macroscopic events, and the whole formalism of QM is nothing more nor less than a recipe for calculating the relative probabilities of such events: it describes nothing at all in the

real world. This viewpoint seems to me free of obvious internal inconsistencies but extremely unpalatable psychologically, in perhaps the same sense as the concept of instantaneous action at a distance seemed unsatisfactory to a minority of 18th-century skeptics (who, of course, were eventually proved right!).

Probably the most widely embraced 'resolution' of the measurement paradox (and certainly the one which has generated the most technical work in this area) is the decoherence point of view, which rests in effect on the denial of stage (3) of the above argument. That is, advocates of this point of view, e.g., (Zurek 1991), argue that:

(a) by the time we get from the level of single electrons, neutrons, etc., to that of cats and Geiger counters, the possibility of exhibiting interference between the relevant (macroscopically distinct) states has indeed vanished,
(b) 'therefore' by that stage one or other outcome has definitely been realized on each individual run of the ensemble.

Unfortunately, almost all the literature in this area (including some work of great technical elegance) has been devoted to embellishing the details of step (a) of the argument, which to my mind was never in serious question (at least as it refers to 'typical' situations: but see below); a minuscule degree of attention, if any at all, is paid to step (b), which to my mind constitutes a severe logical non sequitur: the fact that the evidence against a particular interpretation of the QM formalism has vanished in the transition from micro- to macrolevel does not, in my view, justify us in now embracing that interpretation!

Thus, I believe that no interpretation of QM which takes the latter to be the whole truth about the physical world (including the many variants, such as the consistent-histories interpretation, which I have not discussed explicitly in the last few paragraphs) can provide a satisfactory resolution of the measurement paradox. If that is so, then the only way out would appear to be to question the premiss of stage (2) of the argument, namely that QM is indeed the whole truth; and in particular to explore the hypothesis that for reasons currently unknown to us, the quantum superpositions of macroscopically distinct states which are predicted by the extrapolation of the QM formalism do not in fact occur in nature, but that one or other outcome is always realized. I call such a hypothesis generically macroscopic realism or 'macrorealism'. Various specific proposals for how definite outcomes may be realized have appeared in the literature, the best-developed probably being that associated with the names of Ghirardi, Rimini, Weber and Pearle (GRWP, see, e.g., Pearle et al. 1999). Below I shall not specialize to any particular such proposal but will discuss the question of macrorealistic theories in general.

While the above radical hypothesis may or may not excite the philosophically inclined, its degree of interest to the majority of practicing physicists is likely to depend crucially on the extent to which it can be tested experi-

mentally. And at this stage we appear to face a rather formidable difficulty: If it is indeed true, as postulated in the premiss of step (3) of the argument above, that the quantum interference of macroscopically distinct states (hereafter abbreviated QIMDS) is experimentally unobservable, then by definition all experimental results at this level are guaranteed to come out exactly 'as if' one of the two macroscopically distinct states described by Ψ_1 or Ψ_2 had in fact been realized in each individual run of the ensemble, and thus the question of whether such realization indeed takes place must remain forever inaccessible to experimental resolution. Thus, the only condition under which a test is possible is that the argument for decoherence of macroscopically distinct states is not of universal validity, or to put it in different words, that a correct extrapolation of the QM formalism does not forbid the occurrence of QIMDS under all possible circumstances. It is precisely this hope which has motivated much of the experimental work described in the rest of this paper.

What one would like to do, then, is to set up an experimental situation in which a technically correct application of the formalism of QM leads to the prediction of the QIMDS phenomenon, and to look to see whether this prediction is indeed borne out by the experimental data. If it is not, then one will be led (of course after an appropriately careful check of the details of experiment and QM theory!) to consider seriously the hypothesis that this is evidence for a breakdown of the universal validity of the QM formalism. If on the other hand the QM predictions are found experimentally, then it is of interest to try to rule out, at the relevant level, the alternative hypothesis of macrorealism.

A question arises, of course, as to what exactly one means by 'macroscopically distinct', and how to quantify this concept. I refer to (Leggett 2002) for a detailed discussion of this question, and simply state here my own preferences (prejudices?) for the defining ingredients, namely, a large value of two parameters which I call respectively 'extensive difference' and 'disconnectivity': the former refers to the difference, in suitably chosen 'atomic' units, of expectation values of various extensive quantities in the two states compared, while the latter, at least in the simplest cases, is essentially equivalent to the notion of entanglement as developed in the context of recent work on quantum information; crudely speaking, the degree of disconnectivity is of the order of the lowest-order correlation function one would have to measure in order to directly distinguish the superposition of the two states Ψ_1 and Ψ_2 from a classical mixture of these states. For details see (Leggett 2002) .

One may object at this point that it is totally unrealistic at present, and likely to remain so for the foreseeable future, to measure correlation functions of any order higher than single digits. This is true, but fortunately we do not need to do this; in effect, under appropriate circumstances nature will measure these higher-order correlation functions for us, by implementing the time correlation operator $\hat{U}(t) \equiv \exp(-\mathrm{i}\hat{H}t)$ (see, e.g., Leggett 1984).

6.3 Experiments

I now turn to the various experiments conducted so far which may be regarded as implementing this program, whether by conscious design or not. For a considerably more detailed discussion of all except the most recent experiments, see Sect. 5 of (Leggett 2002).

The experiments which are the most direct analog of the familiar Young's slits experiments with single neutrons are those of the Vienna group (Arndt et al. 1999), in which a Young's slits setup was used to diffract fullerene (C_{60}) molecules. In these experiments the description of the ensemble at the intermediate (two-slit) screen is approximately by a wave function of the general form $a\Psi_1 + b\Psi_2$ where Ψ_1 corresponds to the C_{60} molecule being near slit 1 and Ψ_2 to its being near slit 2. Since the C_{60}^{12} molecule contains 360 electrons and 720 nucleons, both of the quantities Λ and D defined above are of order 1000. Although the evidence for interference is not quite as spectacular as in similar experiments on neutrons, etc., it is fairly clear-cut, and there seems little doubt that it could be made sharper in a new generation of experiments were it thought important to do so .

A historically prior, but more controversial set of experiments which have been interpreted as evidence for QIMDS is on the biological molecule ferritin. In this series of experiments (Harris et al. 1999) the raw data on magnetic rf absorption and noise show evidence of a striking resonance which appears at low temperatures, and this is interpreted as corresponding to the coherent switching of the iron core between opposite directions of magnetization (Julsgaard et al. 2001). Such a coherent oscillation (analogous to the well-known inversion resonance of the NH_3 molecule) requires that the quantum mechanical state at intermediate times should be a linear superposition of the two states; and since the total number of iron atoms in the core is of order 5000, this interpretation would imply values of the two parameters Λ and D of this order. At the time of writing there is an unresolved controversy regarding the correctness of this interpretation of the data: see Sect. 5.2 of (Leggett 2002).

The above two observations of QIMDS rely in effect on the application of the time development operator as indicated above, and the parameters Λ and D are in both cases of the order of the total number N of particles in the relevant system. The next example to be discussed relies on a different principle, and the values of Λ and D are of order $N^{1/2}$ rather than N. This example is a rather spectacular recent experiment (Reid 2000) in quantum optics, which was actually motivated by interest in the nonlocal aspects of QM rather than in the measurement question as such. In this experiment the raw data, if interpreted according to the standard prescriptions of QM, give strong evidence that a degree of entanglement of order $N^{1/2}$ ($\sim 10^6$) was generated between samples each containing $N \sim 10^{12}$ atoms. As explained in Sect. 5 of (Leggett 2002), the data can equally be interpreted as indicating a superposition state with Λ and D values of the same order ($\sim N^{1/2}$). It

should be emphasized that in this case the occurrence of the superposition cannot be read off directly from the raw data, but must be inferred with the help of certain a priori theoretical expectations concerning the description of the system: see (Leggett 2002), loc. cit.

While this is (at least to my knowledge) the only existing experiment which demonstrates QIMDS (at any level which could reasonably be called macroscopic) in a quantum-optical system, there exist other proposals in the literature, e.g., (Reid 2000), which exploit somewhat related considerations: in these cases also, the experimental verification of the phenomenon would not require the use of the time evolution operator, but the values of Λ and D realized would apparently again be of order $N^{1/2}$ rather than N. On the other hand, it is not out of the question that very large values of N (say $\sim 10^{24}$) would be attainable. This state of affairs then raises the question: Is it more significant, from the point of view of the quantum measurement problem, to produce a superposition involving (say) 10^{24} particles where the values of Λ and D are of order 10^{12}, or one in which Λ and D are (say) of order 10^{11} but the total number of particle s in the system is also of this order? In other words, is it the absolute or the relative value of 'disconnectivity' (etc.) which is important? I think that like so many questions relating to possible ways in which QM might break down, this one is legitimately a matter of opinion; for what it is worth, my own prejudice is that it is the relative values which are more significant, at least once one is demonstrably in the 'macroscopic' realm.

I finally turn to the systems in which searches for 'macroscopic quantum effects' have the longest and perhaps richest history, namely superconducting devices based on the Josephson effect. The original foundational motivation for research in this area has been strongly boosted in the last ten years by the hope that systems of this type [or more precisely their collective (macroscopic) degrees of freedom] may be suitable candidates for the elements (qubits) of a quantum computer – a hope which clearly relies on the expectation that these degrees of freedom will behave according to the laws of quantum mechanics. After a long period in which the foundations were firmly laid, progress in this area over the last three years has been spectacular: in 2000 the first circumstantial evidence was attained for the occurrence of QIMDS in systems of this type (with values of Λ and D up to $\sim 10^9$), and in 2002 the first observation was made of real-time oscillations (analogous to those of the NH_3 molecule) between two reasonably ($D \sim 10^4$–10^5) macroscopically distinct states of the magnetic flux. [For details of all of this, see (Leggett 2002), Sects. 5.4 and end of Sect. 6.] As this goes to press, there are reports (Chiorescu 2003) of yet more spectacular observations of such quantum oscillations of the flux between two macroscopically distinct states; from the data given I estimate that the Q-factor is around 400 and the D (or Λ) parameter in the range 10^5–10^6.

6.4 Prospects

Thus, whenever it has been tested, the extrapolation of QM to the macroscopic scale has to date given very impressive agreement with experiment. Where does this leave us? Should we, already at this stage, conclude that QM is indeed the whole truth? I think this would be distinctly premature: while the physical scale (Λ and D values) of the systems in which QIMDS has been verified to occur (in particular, in the superconducting device systems) is already approaching the half-way mark, on a logarithmic scale, between the level of the atom and that of everyday human experience, the systems so far examined are in some sense very crude; they do not have any of the subtleties associated (for example) with biological organization. So while it would no doubt be of some significance, should it turn out to be feasible, to push the limits on Λ and D to values yet closer to the scale of human experience, I believe it would be at least as valuable to explore the continued validity of QM as it were along other dimensions: for example, to attempt to generate quantum superpositions of two states of a molecule such as myoglobin with different biological functionalities, or of the different states of the rhodopsin in the human eye corresponding to perception or non-perception of a weak light signal. Because (inter alia) of the likely severe effect of decoherence under biologically relevant conditions, such experiments will almost certainly be orders of magnitude more demanding than the QIMDS experiments carried out to date, but I believe they would be extremely valuable.

In a different direction, I believe it is important to stress that while existing experiments have very satisfactorily verified the predictions of QM at the level of 'macroscopicness' represented, e.g., by Josephson devices, they have not refuted outright the predictions of the class of 'macrorealistic' theories; it is entirely conceivable (though it might perhaps be thought rather improbable) that some theory of this class would account equally well for existing experimental observations. An experiment which, if it confirms the QM predictions, must automatically refute those of any macrorealistic theory was described some years ago by Garg and the present author (Leggett and Garg 1985); see Sect. 6 of (Leggett 2002) for further references. The large Q-factor obtained for the QIMDS experiment of (Chiorescu 2003) suggests that this experiment is now entirely feasible at the level of superconducting devices, and it is my hope that it will be performed in the near future. If it is done and, as expected by most physicists, confirms the predictions of QM, we shall know for sure that the characteristic weirdness of the quantum world, successfully confined for so many decades to the world of electrons and atoms, has propagated at least a substantial part of the way up towards the level of our direct experience. If the results do not confirm the predictions of QM, then the cat will indeed be among the pigeons!

Acknowledgments

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7 Roundtable Discussion I: Physical Theories, Present and Future

Smolin: I am honored to chair this session. This is an opportunity to say to each other the things that we wanted to say for many years [laughter]. Would one of our philosophers like to open with a conceptual comment?

Saunders: Let me begin with the point of view saying that the universal state is unitarily evolving in time, and our business as physicists is to find interesting patterns in that evolution. These patterns should be explanatory and have interesting stories to tell. One can, of course, find many different patterns which correspond to different consistent history spaces. One can even find patterns where conditional probabilities do not obey the usual rules because they are not even related to a consistent history space at all. But if we do stick to, in particular, a quasi-classical history space, we see patterns which are robust and obey equations of motion that we are familiar with. Such a propagating pattern is exactly the sort of thing that we are constituted of. But on that sort of a view, call it view (1), every history in the quasi-classical domain is real – they all exist. They differ only in their probability amplitudes. These amplitudes are also a part of the reality, and the fact that they all coexist is not in contradiction with the fact that one can decompose the universal state in an entirely different basis, because it is the superposition that ultimately makes up that universal state. All of that makes for a package which I think is coherent and intelligible, although there is a remaining outstanding problem of what we mean by 'probability'. It's a difficult question. One can choose a different point of view altogether which gives a familiar and perhaps more simple answer.

The different point of view, view (2), is that there is a unique history, which is real, while all the rest are not real. One then analyses the universal state in order to get a probability measure over histories, and a single history is realized.

Is there some view (3)? I think there is one, which Griffiths[1] subscribes to: For *every* history space, there is one (but only one) history that is real. Again, for each history space, one has a familiar notion of probability.

[1] R.B. Griffiths: Choice of consistent family, and quantum incompatibility, Phys. Rev. A **57**, 1604–1618 (1998).

And I would now add view (4), the version of the many-worlds interpretation in which there is a unique consistent history space. All of the histories are realized, one of them being ours, and the probability is purely epistemic – we just don't know which history is ours; we know what it is up to the present time but we don't know what it is in the future. So we have epistemic probability and we have a probability measure on this space, interpreted in terms of our subjective ignorance of what to expect in the future.

In view (1), the fundamental object is the universal state, and there are many interesting ways of showing patterns within that universal state. The use of consistent histories is a very useful way. One then has no interference among histories. Quasi-classical histories are even better, because then you have approximate equations of motion that you can use. But the fundamental object is the superposition of all such histories. 'Reality' is the universal state, which is propagating unitarily, and of course we, our environment, and so on, are a small part of this universal state. One can make a distinction between what is actual relative to us, and what is not, which of course links back to Everett's original idea of 'relative states'. It then turns out that we are correlated with a unique past, and of course that we are not correlated with an alternative state of affairs in which we are not meeting here in this room. Option (1) has the familiar branching tree-like picture.

I think the notion of probability is intelligible and meaningful in option (1), but it is much more familiar in options (2), (3), and (4). But on any of the latter, you place great stress on what precisely is the history space. On option (3), you are saying that one history is realized from every history space, but it is the history space which ours belongs to that has the relevant probability measure, and one needs to know what it is. On options (2) and (3) there is something like reduction of the wave function. That obviously puts great weight on the question of what the history space is. People like GRW,[2] or those who work on the pilot wave theory [see Hiley in Chap. 16], spend years struggling courageously to say precisely what, in effect, the history space is. But even on option (4) the same applies. You are having all of the histories realized in a unique history space but still one has to know what that unique history space is. And once you say that you might as well let go of all of the other histories, and just stick with one of them, so I think option (4) more or less collapses down to option (2). That leaves option (1) as the only one that does not require any precise specification of the history space, because the entire point is that the use of the history space is instrumental. One is extracting interesting patterns. They are all there. But the price is that there is no longer an easy and familiar answer to the question 'What is probability?'

Hartle: I start from the idea that that quantum theory predicts probabilities for alternative histories of the universe. I understand what these probabilities

[2] G. Ghirardi, A. Rimini, and T. Weber: Unified dynamics for microscopic and macroscopic systems, Phys. Rev. D **34**, 470–479 (1986).

mean through understanding how they are used in physics. Namely, we use the theory to predict probabilities and check the theory against those predictions that have a probability near 1 or 0. Predicted probabilities include those conditioned just on the initial state of the universe used largely in cosmology, but also probabilities conditioned on further information (conditional probabilities). An example is the probability of the sun coming up tomorrow at a particular time, which is conditioned on the state of the universe but also on the information we have from our earlier observations of the solar system. The usual statistical interpretation of probabilities follows when this understanding is applied to a very large ensemble of identical situations in the universe.

With this view of how to use quantum theory, I'm less compelled than some of you to take a stand on 'what is realized'. I would like to interpret all statements about the theory in terms of the probabilities it supplies. It seems to me that a statement about 'what is realized' is an extra feature that can be added, perhaps to clarify probability, perhaps for other purposes that make people happy. What I can say is that quantum theory does not distinguish between different decohering sets except by properties such as classicality. Therefore one can't adopt position (4) that there is a unique set of decoherent histories that is the quasi-classical realm.

Dürr: The basic entity is the operator algebra. The probability aspect enters only when we measure. Then we deal with particular representations of this algebra in a linear state space, which depend on how we split the 'Whole' (the Cosmos) into the smaller subsystem under observation and the huge and infinite remainder, the 'background' system containing the observer and his measuring devices, all treated classically. The particular split chosen is a matter of convenience, depending on what questions we ask and want to have answered. Different choices of such a separation define different approximations. The wave functions usually considered are features of the non-relativistic quantum mechanics. They lose their intelligible meaning in a relativistic quantum theory which necessarily requires 'many-body' states or representations of quantum fields. The 3-space locations are no longer operators but simply parameters like the time. In addition, the quantization conditions are essentially (or even exclusively) based on 'non-classical' spinors. The probability aspect only becomes meaningful if one can approximately define asymptotic states consisting of free particles. Mathematically, the probability features are related to the metric structure of the state space which, in general, will not be a Hilbert space (positive definite metric) allowing a (positive definite) probability interpretation, but a Nevanlinna space (indefinite metric) characteristic, in particular, of gauge-type field theories.

Hartle: You're certainly right that we make approximations in everyday calculations such as dividing the universe into two parts to describe a mea-

surement situation. But in quantum cosmology, we consider the state of the universe as a whole and from it make predictions for the probabilities of the behavior of the universe as a whole. We can't assume ignorance of this state and replace it by a unit density matrix. That would imply an infinite temperature and other features that are in contrary to our experience.

Dürr: Near the big bang, the 'state' certainly has to be considered to be 'quantum'. I just cannot see how our experience with non-relativistic quantum theories employing 'quasi-classical history' notions and other classical 'egg shells' are at all appropriate for tackling this problem. I have no clue how to write down expressions which would simulate something like the probability amplitude of the 'Whole'. In this case the observed system would be the Whole and there is nothing left for the 'remainder' containing the observer.

Hartle: Nevertheless we can investigate the quantum state of the whole universe. If quantum mechanics applies to the whole universe then every prediction is fundamentally a quantum mechanical probability. Especially in the late universe, as you say, the probabilities may be high that some variables behave approximately classically but that classical behavior depends crucially on the quantum state.

Rovelli: Let's say I use the conditional model to calculate probabilities: "Fact A has happened, then ...." However, what is the state of this 'Fact A' in the theory? Are all the physicists going to agree that that fact really happened?

Hartle: Questions about 'what happened' are answered by probabilities about the past in quantum mechanics. In quantum cosmology, from a present data plus the initial state of the universe you can calculate the probabilities for alternatives in the past. (In classical physics you need only the present data to calculate probabilities of the past, but in quantum mechanics you also need the initial state.) If future aliens recover phases that we have lost, they could be unable to make the retrodictions about the past that we can. The past is not permanent in this sense. The permanency of the past is generally an approximate notion, but in many familiar situations a truly excellent approximation.

Rovelli: So something which for me is a fact for me in the present, for someone else might not be a fact in the future?

Hartle: Yes, it's somewhat like the situation in a Stern–Gerlach experiment. When the beam is split you can infer the past spin of the particle from the present position of the beam. But if the beams are coherently recombined,

you can't make those inferences.

't Hooft: There are different questions that one can ask about reality. The first question is: what are the laws of physics, what are the degrees of freedom, and what was the initial state of the universe? The other question is: given a certain system, what can we say about it, about reality? A third kind: given a system, what can we predict about it in the future? A lot of data is missing. We don't know all of the rules of physics, we don't know the exact initial state of the universe, only approximately, we don't know the exact ontological nature of the world, and we can't make good predictions about the future. So we have three kinds of ignorance, and we must keep them distinct.

Stapp: Jim [Hartle], in your famous paper with Gell-Mann,[3] you took some Everettian idea, with Griffiths'[4] consistent history, as well as decoherence.[5] So you used three ideas together. But essentially, you are Everettian (camp 1 as Simon [Saunders] said). Yet you said that we calculate probabilities and make predictions – more like the Copenhagen view. Are you sticking to the Everettian view of a wave function of the universe, or do you admit that we can't know the universal state, and stick to conditional probabilities?

Hartle: Copenhagen quantum mechanics is included in a quantum mechanics that deals with the wave function of the whole universe. Copenhagen quantum mechanics is an approximation to that more general framework that is appropriate for measurement situations. If you have a measurement situation within the universe, calculations based on an approximate wave function of a measured subsystem will give probabilities that coincide with a calculation based on the wave function of the whole universe.

Stapp: Is it your position that there is a reality out there which is described by the wave function of the universe?

Hartle: To answer your question, Henry, I have to say a little about several different meanings of 'reality'. First there is the reality of everyday experience – the agreement among most human observers about what is happening and

[3] M. Gell-Mann and J.B. Hartle: In: *Complexity, Entropy, and the Physics of Information*, SFI Studies in the Sciences of Complexity, Vol. VIII, ed. by W. Zurek, Addison Wesley, Reading, MA, or in: *Proceedings of the 3rd International Symposium on the Foundations of Quantum Mechanics in the Light of New Technology*, ed. by S. Kobayashi, H. Ezawa, Y. Murayama, and S. Nomura, Physical Society of Japan, Tokyo (1990).

[4] R. Griffiths: Consistent histories and the interpretation of quantum mechanics, J. Stat. Phys. **36**, 219 (1984).

[5] D. Giulini. E. Joos, C. Kieffer, J. Kupsch. I.O. Stamatesscu, and H.D. Zeh (Eds.): *Decoherence and the Appearance of a Classical World in Quantum Theory*, Springer (1996).

has happened. Humans who disagree, as those with certain mental illnesses, are said to be 'out of touch with reality'. This agreement is explained in quantum mechanics by the fact that the great majority of us are all using various coarse-grainings of one set of histories constituting the quasi-classical realm. That is a human specific notion of reality.

But many scientists and philosophers are interested in a different notion of reality – roughly 'what's out there *independent* of human cognition'. Quantum reality in this sense is very different from our everyday notion because there are many sets of decoherent histories that are different from the quasi-classical realm. These sets of coarse-grained histories are generally *complementary*. That is, there is no finer-grained decoherent set of histories that contains them all. The theory does not distinguish between these, except by criteria such as classicality. In particular, it does not say that one set is real and the others are not. Quantum reality therefore consists of one history in each distinct decoherent set of possible histories.

So yes, Henry, I would say that there *is* 'reality out there', both the everyday notion of reality and the general quantum reality based on many complementary decoherent sets of histories.

Saunders: There has been a lot of talk here about 'reality'. This is a philosophical term that has many readings. The more clear-cut problem is that we don't know where to go with quantum gravity. The crucial question is whether we are going to look at the problem of measurement as something that gives us a research strategy or not. Is it going to fit into what we are going to do in quantum gravity? If one is going to reject the Everett interpretation, one is then forced to look for a precise definition of the history space. That seems to require physical principles which will be relevant to quantum gravity. Dirac is famous for saying that the problem of measurement is not really interesting because it doesn't give us any new equations, and it is the equations that matter. But how wrong can you get? We have two very clear proposed solutions for the problem of measurement . One is the de Broglie–Bohm, the other GRW. In both cases you have new equations!

Zeilinger: We now have a situation in which we have a number of different viewpoints of quantum mechanics and we violently disagree! I say that there is no measurement problem at all. For me this is actually a virtue of the whole business while for others it's a big problem. I'm talking strictly about the interpretation. All interpretations are equivalent (GRW is an exception). It all depends on whether any one of these different ways of looking at it really leads to a new view of the world. And that could very well be what I view as the Copenhagen interpretation and its strict minimalist program: Do not assume anything you cannot strictly talk about, verify experimentally and so on. Or it could be another program, we don't know. We simply have to try all of them. I think we should leave all these paths open because we

don't know where and when a clever young guy will build some new insights.

Saunders: It seems to me to be widely doubted that there is even one coherent, intelligible interpretation that leaves quantum mechanics alone. I think the fundamental question to resolve is: Do we have even one conservative interpretation of quantum mechanics?

Zeilinger: I do not say that your interpretation is not coherent or not intelligible. I just don't like it [laughter]! No, seriously, it comes down to a matter of personal choice! I don't think that there is more to that. I would never claim that what you say is not intelligent, not consistent, I would never claim that! I just don't like it for some emotional reasons. This is all that can distinguish between the different points of view. And I hope that this point of view is also not incoherent [laughing].

Hartle: Are you saying that different interpretations are a matter of viewpoint rather than experiment?

Zeilinger: In this respect there are no differences between 'many worlds', Bohm's and my own viewpoint.

Unidentified: What about GRW?

Zeilinger: GRW is a different theory. I'm not talking about alternative theories.

Elitzur: Anton, suppose that instead of a Schrödinger cat, there is a physicist. Will she have all the information about what went on with her during the time she was in superposition? Suppose you could fulfill your dream, sending Greenberger through a wall with a double slit: What will he say?

Zeilinger: If he remembers whether he took the right or the left path, there will be no quantum interference. But if he forgets, that is, 'forgets' in a quantum erasure way, then it's ok.

Elitzur: That's actually the meaning of interference, namely, erasure, isn't it? And it won't make any difference whether it's a particle or a physicist?

Zeilinger: Right. I see no reason to claim that quantum interference with a living object, including humans, might not work some day! For Schrödinger's cat, it is very often said that we will not be able to see interference phenomena for a living object because the living object needs to interact with the environment all the time. But I can see no reason why, for a very small living organism, we should not be able to put it in a tiny container with all the life

support systems, so we have no interaction with the environment. And then you can have the whole system in a superposition and eventually observe interference.

Smolin: Does anyone have something nice to say as a closing remark?

Zeilinger: I guess we would all be very surprised if we weren't very surprised by the future development of the field!

8 Determinism Beneath Quantum Mechanics

Gerard 't Hooft

Contrary to common belief, it is not difficult to construct deterministic models where stochastic behavior is correctly described by quantum mechanical amplitudes, in precise accordance with the Copenhagen–Bohr–Born doctrine. What is difficult, however, is to obtain a Hamiltonian that is bounded from below, and whose ground state is a vacuum that exhibits complicated vacuum fluctuations, as in the real world.

Beneath quantum mechanics, there may be a deterministic theory with (local) information loss. This may lead to a sufficiently complex vacuum state, and to an apparent non-locality in the relation between the deterministic ('ontological') states and the quantum states, of the kind needed to explain away the Bell inequalities.

Theories of this kind would not only be appealing from a philosophical point of view, but may also be essential for understanding causality at Planckian distance scales.

8.1 Motivation

The need for an improved understanding of what quantum mechanics really is need hardly be explained in this meeting. My primary concern is that quantum mechanics, in its present state, appears to be mysterious. It should always be the scientist's aim to take away the mystery of things. It is my suspicion that there should exist a quite logical explanation for the fact that we need to describe probabilities in this world quantum mechanically. This explanation can presumably be found in the fabric of the laws of physics at the Planck scale.

However, if our only problem with quantum mechanics were our desire to demystify it, then we could put forward the fact that, as it stands, quantum mechanics works impeccably. It predicts the outcome of any conceivable experiment, apart from some random ingredient. This randomness is perfect. There never has been any indication that there would be any way to predict where in its quantum probability curve an event would actually be detected. Why not be at peace with this situation?

One answer to this is quantum gravity. Attempts to reconcile general relativity with quantum mechanics lead to a jungle of complexity that is

difficult or impossible to interpret physically. In a combined theory, we no longer see 'states' that evolve with 'time', we do not know how to identify the vacuum state, and so on. What we need instead is a unique theory that not only accounts for quantum mechanics together with general relativity, but also explains how matter behaves. We should find clues pointing towards the correct unifying theory underlying the standard model, towards explanations of the presumed occurrence of supersymmetry, as well as the mechanism(s) that break it. We suspect that deeper insights into what and why quantum mechanics is should help us to further understand these issues.

Related to the question of quantizing gravity is the problem of quantizing cosmology. Astrophysicists tell us that the Universe started with a 'Big Bang', but, at least at first sight, such a statement appears to be at odds with the notions of quantum mechanical uncertainty. In principle, we could know the state the Universe is in at present, and then solve the Schrödinger equation backwards in time, but this should yield a quantum superposition of many configurations, not just a Big Bang. Questions of this sort may seem of purely academic nature, but they become very concrete as soon as one attempts to construct some reasonable model for a 'quantum universe'. The notion of a quantum state of the Universe appears to defy logic.

Attempts have nevertheless been made to reconcile quantum mechanics with cosmology. Whether the proposed schemes may be viewed as a satisfactory picture of our world is difficult to discuss. To convince someone that they are flawed may be as difficult as changing someone's religious beliefs. Therefore, I shall refrain from trying to do this; instead, one further issue will be displayed in the next section.

8.2 Holography

Black holes are not only legitimate solutions of Einstein's field equations for the gravitational force, but one can also show quite easily that black holes inevitably form under given favorable initial conditions of conventional matter configurations. Such 'conventional' black holes are very big, having a radius at least of the order of 10 km. Therefore, they are usually considered as classical, i.e., non-quantum mechanical objects. But, at least in principle, they should also obey the laws of quantum mechanics. Elementary particles in the vicinity of a black hole should be described by quantum field theory, and the laws of general relativity should dictate how to handle quantum field theory here. As was shown by S. Hawking, this exercise leads to the astonishing result that particles must emerge from a black hole [1].

Mathematically, the explanation for this effect is that time is measured by freely falling observers in a coordinate frame that is fundamentally different from the coordinate frame used by the onlooking observer outside the black hole. Physically, one may explain the emission as a gravitational tunneling effect, comparable to the pair creation of charged particles in the presence

of a strong electric field. The emission rate is precisely computable, and conventional theory gives a flux of particles corresponding to a temperature

$$T_{\rm H} = \frac{\hbar c^3}{8\pi k G M_{\rm BH}} \,, \tag{8.1}$$

where k is Boltzmann's constant and $M_{\rm BH}$ is the mass of the black hole.

Hawking's result can be used to estimate the density of quantum states of a black hole. Assuming a transition amplitude $\mathcal{T}_{\rm in}$ for the absorption process, there are two ways to write an estimate for the absorption cross-section $\sigma(\boldsymbol{k})$ for an amount of matter δE with momentum $\boldsymbol{k}$ by a black hole of mass $M_{\rm BH}$:

$$\sigma \approx 2\pi r_+^2 = 8\pi M_{\rm BH}^2 \,, \tag{8.2}$$

$$\sigma = |\mathcal{T}_{\rm in}|^2 \rho(M_{\rm BH} + \delta E)/v \,. \tag{8.3}$$

Here, r_+ is the radius of the outer event horizon, $\rho(M)$ is the density of states of a black hole with mass M, and v is the velocity of the absorbed particle. The probability $W\,{\rm d}t$ of particle emission during a time interval ${\rm d}t$ can also be written in two ways:

$$W\,{\rm d}t = |\mathcal{T}_{\rm out}|^2 \rho(M_{\rm BH})\,{\rm d}t/V \,, \tag{8.4}$$

$$W\,{\rm d}t = \frac{\sigma(\boldsymbol{k})v}{V} {\rm e}^{-\delta E/kT_{\rm H}} {\rm d}t \,. \tag{8.5}$$

Here, V is the volume of a box, in which the wave function of the emitted particle is normalized. Dividing (8.2)–(8.5), we get, in Planck units [2]

$$\frac{\rho(M+\delta E)}{\rho(M)} = \frac{|\mathcal{T}_{\rm out}|^2}{|\mathcal{T}_{\rm in}|^2}\, {\rm e}^{\delta E/kT_{\rm H}} = {\rm e}^{8\pi M \delta E} \,. \tag{8.6}$$

We have assumed here that $|\mathcal{T}_{\rm out}| = |\mathcal{T}_{\rm in}|$. All that is needed for this assumption is PCT invariance, since $\sigma(\boldsymbol{k})$ is symmetric under P and C. For all known field theories, PCT is a perfect symmetry. Needless to say, we do not know this to be so for quantum gravity, but it would be a natural assumption.

Equation (8.6) is to be seen as a differential equation that is easily integrated to give

$$\rho(M) = {\rm e}^{4\pi M^2 + \mathcal{C}} = \mathcal{C}' 2^{A/A_0} \,, \tag{8.7}$$

where $\mathcal{C}$ and $\mathcal{C}'$ are integration constants, $A = 4\pi r_+^2$ is the black hole area, and $A_0 = 4\ln 2$ in Planck units. One concludes that the density of quantum states of a black hole is that of an object with A/A_0 free Boolean parameters on its surface. The integration constant represents a fixed degree of freedom that all black holes have in common. The result (8.7) can also be derived using thermodynamics, but then one has to cope with the difficulty that black holes, embedded in a thermal environment, are unstable because of their negative specific heat (since they cool off when energy is added to them) [3].

In one respect, this result appears to be quite interesting and acceptable. Apparently, the quantum states of a black hole form a discrete set, just as if the black hole were a fairly ordinary object, easily to become macroscopic, in the astronomical case. Black hole formation and evaporation can indeed be described in terms of quantum amplitudes, and if the black hole is very tiny, these amplitudes can be represented in Feynman diagrams.

On closer inspection, however, there are several problems with this result. We would have thought that a general coordinate transformation transforms states into states. An ingoing observer describes what (s)he sees in terms of particle states superimposed on an approximately flat spacetime environment, using regular coordinates. The outside observer uses the black hole coordinates featuring a horizon. The states observed by the outside observer are counted by discrete variables of one bit for every area unit A_0. How can this mapping of discrete states onto the continuum of states for the ingoing observer be unitary? How does the ingoing observer count his/her states? We should have expected the number of these states to scale with the bulk volume of his environment, not with the area.

Hawking's calculation gives no clues here. On the contrary, it appears to tell us that, even if the initial state of an imploding object were a quantum mechanically pure state, the radiating black hole that emerges after some time would nevertheless be in a quantum mechanically mixed state [4]. Such a transition cannot be described by any Schrödinger equation. Does the black hole, viewed as an isolated object, disobey the quantum code? This is what was concluded initially, but most of us now agree that such a conclusion must be premature [2].

If, on the other hand, information is conserved in unitary evolution equations, how is it that the information in the ingoing particles is transmitted to the outgoing ones?

A first clue towards answering this question was provided by taking into account the fact that ingoing particles interact with outgoing ones when they pass each other. The gravitational interaction here diverges. Early ingoing particles meet late outgoing ones in an entirely different local Lorentz frame, so that the relative energy, that is, the energy in the center of mass frame, is large, and this number diverges exponentially with the time difference of the two particles [5]. Taking this into account, one does find a unitary scattering matrix relating outgoing particles to ingoing ones, but the spectrum of states does not seem to be bounded by the horizon area. Such a bound presumably has to come from the transverse components of the gravitational interactions, which are much harder to calculate [6].

Requiring that the number of states in some region of space, described by a theory, is bounded by the surface area of this region, seems to be paradoxical. This paradox seems to be as deep and fundamental as the one that lead M. Planck to his postulates of quantum mechanics, or, in other words, we

expect its resolution to involve a paradigm shift. How can we have locality in three-space, but numbers bound by two-space?

In certain versions of string theory, these apparently conflicting demands are met to some extent [7], except that the concept of locality appears to be ignored. The amount of 'magic' required for these ideas to work is still not acceptable. It is this author's belief that the true reason for the mysterious nature of this problem is our insistence on sticking to the language of quantum mechanics. It seems to be only natural to see a link between the mysteries of string theory and those of the correct interpretation of quantum mechanics.

8.3 Harmonic Oscillators

It is instructive to ask how a deterministic system can be addressed using the mathematics of quantum mechanics. Our starting point is that we may have simple autonomous dynamical systems, where later we will decide how they should be coupled. Thus, we start with a deterministic system consisting of a set of N states [8]

$$\big\{(0),(1),\ldots,(N-1)\big\}\,, \tag{8.8}$$

on a circle. Time is discrete, the unit time steps having length τ (the continuum limit is left for later). The evolution law is

$$t \to t+\tau: \qquad (\nu) \to (\nu+1 \mod N)\,. \tag{8.9}$$

Introducing a basis for a Hilbert space spanned by the states (ν), the evolution operator can be written as

$$U(\Delta t=\tau) = \mathrm{e}^{-\mathrm{i}H\tau} = \mathrm{e}^{-\pi\mathrm{i}/N} \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}. \tag{8.10}$$

The phase factor in front of the matrix is of little importance; it is there for future convenience. Its eigenstates are denoted by $|n\rangle$, $n=0,\ldots,N-1$. They are found to be

$$|n\rangle = \frac{1}{\sqrt{N}} \sum_{\nu=1}^{N} \mathrm{e}^{2\pi\mathrm{i}n\nu/N}(\nu)\,, \qquad n=0,\ldots,N-1\,. \tag{8.11}$$

This law can be represented by a Hamiltonian using the notation of quantum physics:

$$H|n\rangle = \frac{2\pi(n+1/2)}{N\tau}|n\rangle\,. \tag{8.12}$$

The term $+1/2$ comes from the aforementioned phase factor. Next, we apply the algebra of the $SU(2)$ generators L_x, L_y and L_z, whence

$$N \stackrel{\text{def}}{=} 2\ell+1\ , \qquad n \stackrel{\text{def}}{=} m+\ell\ , \qquad m=-\ell,\dots,\ell\ . \tag{8.13}$$

Using the quantum numbers m rather than n to denote the eigenstates, we have

$$H|m\rangle = \frac{2\pi(m+\ell+1/2)}{(2\ell+1)\tau}|m\rangle \qquad \text{or} \qquad H=\frac{2\pi}{(2\ell+1)\tau}(L_z+\ell+1/2)\ . \tag{8.14}$$

This Hamiltonian resembles the harmonic oscillator Hamiltonian when $\omega = 2\pi/(2\ell+1)\tau$, except for the fact that there is an upper bound for the energy. This upper bound disappears in the continuum limit, if $\ell\to\infty$, $\tau\downarrow 0$. Using L_x and L_y, we can make the correspondence more explicit. Write

$$L_\pm|m\rangle \stackrel{\text{def}}{=} \sqrt{\ell(\ell+1)-m(m\pm1)}|m\pm1\rangle\ , \tag{8.15}$$

$$L_\pm \stackrel{\text{def}}{=} L_x \pm \mathrm{i}L_y\ , \qquad [L_i,L_j]=\mathrm{i}\epsilon_{ijk}L_k\ , \tag{8.16}$$

and define

$$\hat{x}\stackrel{\text{def}}{=}\alpha L_x\ , \qquad \hat{p}\stackrel{\text{def}}{=}\beta L_y\ , \qquad \alpha\stackrel{\text{def}}{=}\sqrt{\frac{\tau}{\pi}}\ , \qquad \beta\stackrel{\text{def}}{=}-\frac{2}{2\ell+1}\sqrt{\frac{\pi}{\tau}}\ . \tag{8.17}$$

The commutation rules are

$$[\hat{x},\hat{p}]=\alpha\beta\mathrm{i}L_z=\mathrm{i}\left(1-\frac{\tau}{\pi}H\right)\ , \tag{8.18}$$

and since

$$L_x^2+L_y^2+L_z^2=\ell(\ell+1)\ , \tag{8.19}$$

we have

$$H=\frac{1}{2}\omega^2\hat{x}^2+\frac{1}{2}\hat{p}^2+\frac{\tau}{2\pi}\left(\frac{\omega^2}{4}+H^2\right)\ . \tag{8.20}$$

Now consider the continuum limit $\tau\downarrow 0$, with $\omega=2\pi/(2\ell+1)\tau$ fixed, for those states for which the energy stays limited. We see that the commutation rule (8.18) for $\hat{x}$ and $\hat{p}$ becomes the conventional one, and the Hamiltonian becomes that of the conventional harmonic oscillator:

$$[\hat{x},\hat{p}]\to\mathrm{i}\ , \qquad H\to\frac{1}{2}\omega^2\hat{x}^2+\frac{1}{2}\hat{p}^2\ . \tag{8.21}$$

There are no other states than the legal ones, and their energies are bounded, as can be seen, not only from (8.20), but also from the original definition (8.14). Note that, in the continuum limit, both $\hat{x}$ and $\hat{p}$ become continuous operators, since both α and β tend to zero.

The way in which these operators act on the 'primordial' or 'ontological' states (ν) of (8.9) can be derived from (8.15) and (8.17), if we realize that the states $|m\rangle$ are just the discrete Fourier transforms of the states (ν) [see (8.11)]. This way, the relation between the eigenstates of $\hat{x}$ and $\hat{p}$ and the states (ν) can also be determined. Only in a fairly crude way, $\hat{x}$ and $\hat{p}$ give information about where on the circle our ontological object is; both $\hat{x}$ and $\hat{p}$ narrow down the value ν of our states (ν).

The most important conclusion from this section is that there is a close relationship between the quantum harmonic oscillator and the classical particle moving around a circle. The period of the oscillator is equal to the period of the trajectory along the circle. We started our considerations by having time discrete, and only a finite number of states. This is because the continuum limit is a rather delicate one. One cannot start directly with the continuum because the Hamiltonian does not then seem to be bounded from below.

The price we pay for a properly bounded Hamiltonian is the square root in (8.15). It may cause complications when we attempt to introduce interactions, a problem that has not yet been properly worked out.

Starting from this description of harmonic oscillators in terms of deterministic models, one may attempt to construct deterministic theories describing, for instance, free bosonic particles [9]. Strings can also be seen as collections of harmonic oscillators. A first attempt to write string theory in deterministic terms failed because conformal invariance could not be built in [11]. Apparently, further new ideas are needed here.

8.4 Continuous Degrees of Freedom

In the previous section, a discrete, periodic system was considered and we took the continuum limit in the end. Could one not have started with a continuous model right from the beginning?

Take a Newtonian equation

$$\frac{\mathrm{d}}{\mathrm{d}t} q^i(t) = f^i(\boldsymbol{q}) \; . \tag{8.22}$$

We can write the quantum Hamiltonian

$$H = \sum_i p_i f^i(\boldsymbol{q}) \; , \qquad p_i = \frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial q^i} \; . \tag{8.23}$$

This is quantum language for a classical, deterministic system. It works because the Hamiltonian is linear, not quadratic, in the momenta p_i. The difficulty linking this with real quantum mechanics is that this Hamiltonian cannot possibly be bounded from below, so that there is no ground state.

8.5 Massless, Non-Interacting Fermions

Massless, non-interacting fermions are entirely deterministic. this can be demonstrated by identifying the 'beables' for this system. Beables are a complete set of observables $\mathcal{O}_i(t)$ that commute at all times:

$$[\mathcal{O}_i(t), \mathcal{O}_j(t')] = 0 \ , \quad \forall \ t, t' \ . \tag{8.24}$$

To begin with, consider only first-quantized, chiral fermions. They have a two-component complex wave function obeying the Hamilton equation for

$$H = \boldsymbol{\sigma} \cdot \boldsymbol{p} \ , \qquad \sigma_i \, \sigma_j = \delta_{ij}\mathbb{I} + \mathrm{i}\epsilon_{ijk}\sigma_k \ , \tag{8.25}$$

where σ_i are the Pauli matrices. Consider the set

$$\mathcal{O}_i(t) = \left\{ \hat{\boldsymbol{p}} \, , \ \hat{\boldsymbol{p}}{\cdot}\boldsymbol{\sigma} \, , \ \hat{\boldsymbol{p}}{\cdot}\boldsymbol{x} \right\} \ , \tag{8.26}$$

where

$$\hat{p}_i \equiv \pm \frac{p_i}{|p|} \ , \qquad \hat{p}_x > 0 \ . \tag{8.27}$$

These operators obey closed time evolution equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{\sigma}(t) \ , \qquad \frac{\mathrm{d}}{\mathrm{d}t}\big(\hat{\boldsymbol{p}}{\cdot}\boldsymbol{x}(t)\big) = \hat{\boldsymbol{p}}{\cdot}\boldsymbol{\sigma} \ , \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{p}}{\cdot}\boldsymbol{\sigma}\right) = 0 \ . \tag{8.28}$$

$$\hat{\boldsymbol{p}}(t) = \hat{\boldsymbol{p}}(0) \ , \qquad \hat{\boldsymbol{p}}{\cdot}\boldsymbol{\sigma}(t) = \hat{\boldsymbol{p}}{\cdot}\boldsymbol{\sigma}(0) \ , \qquad \hat{\boldsymbol{p}}{\cdot}\boldsymbol{x}(t) = \hat{\boldsymbol{p}}{\cdot}\boldsymbol{x}(0) + \hat{\boldsymbol{p}}{\cdot}\boldsymbol{\sigma}(0)t \ . \tag{8.29}$$

The fact that all operators in (8.26) commute with one another is easy to establish, with the possible exception of $[\hat{p}_i, \, \hat{\boldsymbol{p}}{\cdot}\boldsymbol{x}]$. The fact that the latter vanishes is most easily established in momentum space, realizing that $\boldsymbol{p} \cdot \boldsymbol{x}$ is the dilatation operator, while $\hat{\boldsymbol{p}}$ keeps the same length 1 under dilatations:

$$[\hat{\boldsymbol{p}}{\cdot}\boldsymbol{x}, \, \hat{p}_i] = \mathrm{i}\left(\hat{\boldsymbol{p}}{\cdot}\frac{\partial}{\partial \boldsymbol{p}}\right)\hat{p}_i = 0 \ . \tag{8.30}$$

The physical interpretation of this result is that the dynamical behaviour of a massless, chiral, non-interacting fermion is exactly like that of an infinite, flat, oriented sheet, moving with the speed of light in a direction orthogonal to the sheet. $\pm\hat{p}$ gives the direction of the sheet, $\hat{\boldsymbol{p}}{\cdot}\boldsymbol{\sigma}$ gives the sign of its orientation and $\hat{\boldsymbol{p}}{\cdot}\boldsymbol{x}$ its distance from the origin.

As before, we encounter the difficulty that, in this deterministic system, the Hamiltonian is not bounded below, again because it is linear in the momenta p_i. Thus, there exists no ground state. In this case, however, P.A.M. Dirac told us what to do: second quantization. Assume an indefinite number of particles with Hamiltonian (8.25). Consider the range of energies they can have. Take the state where all negative energy states are occupied

by a particle, all positive energy states are empty. That is the state with lowest possible total energy, the vacuum state. It is standard procedure, but it does require our particles to obey Fermi statistics, or, in other words, no two particles are allowed in the same state, and interchanging two particles does not change a state into a different one.

The latter condition is easily met, but to forbid two particles to be in the same state requires some sort of repulsion. The easiest procedure is to have at each value for the unit orientation vector $\hat{\boldsymbol{p}}$ a grid with some finite spacing a. No two sheets are allowed on the same lattice point. Then we can count the states exactly as in a fermion theory and the second quantization procedure works. The limit $a \downarrow 0$ can be taken without any difficulty.

8.6 Locality

Let us focus a bit more on the ontological states for massless fermions. They are characterized by an orientation $\hat{\boldsymbol{k}}$ (obeying $|\hat{\boldsymbol{k}}| = 1$), and a distance scalar z. Furthermore, we need the quantum operator conjugate to z, which we call $q \equiv -\mathrm{i}\partial/\partial z$. We define η_3 to be the sign of q, and $\hat{\boldsymbol{k}}$ is the orientation of the sheet with its sign chosen so that it moves in the positive $\hat{\boldsymbol{k}}$ direction:

$$(\hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma})\hat{\boldsymbol{p}} \equiv \hat{\boldsymbol{k}} \equiv \eta_3 \frac{\boldsymbol{p}}{|\boldsymbol{p}|} , \qquad \eta_3 \equiv \frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{|\boldsymbol{p}|} = \pm 1 . \tag{8.31}$$

$z = \hat{\boldsymbol{k}}\cdot\boldsymbol{x}$ is the distance of the sheet from the origin, apart from its sign, which denotes whether the sheet moves away from or towards the origin. To define the original components of the vector $\boldsymbol{p}$, we first have to find its length $|\boldsymbol{p}|$. This we take to be the operator

$$|\boldsymbol{p}| \equiv |q| \equiv -\mathrm{i}\eta_3 \frac{\partial}{\partial z} , \qquad \text{so} \quad \boldsymbol{p} = -\mathrm{i}\hat{\boldsymbol{k}} \frac{\partial}{\partial z} . \tag{8.32}$$

$\hat{\boldsymbol{k}}$ and z are the ontological variables, or beables, whereas q and η_3 are changeables. We have $H = q$, so the dynamical equations are now simply

$$\dot{z} = 1 , \qquad \dot{\hat{\boldsymbol{k}}} = 0 , \tag{8.33}$$

which are the equations of a sheet moving in a fixed direction. Since (8.32) defines the momenta, and their canonically conjugate operators the positions, we should now be in a position to compute the conventional wave function $\psi(\boldsymbol{x}, \sigma_3)$, $\sigma_3 = \pm 1$, if we have some wave function $\psi(\hat{\boldsymbol{k}}, z)$. A fairly delicate calculation gives

$$\psi(\boldsymbol{x}, \pm) = \frac{1}{2\pi} \boldsymbol{\partial}_x^2 \int \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \begin{pmatrix} \cos\frac{1}{2}\theta \\ \mathrm{e}^{-\mathrm{i}\phi} \sin\frac{1}{2}\theta \end{pmatrix} |\hat{k}, z\rangle , \tag{8.34}$$

where we have used the notation

$$z = \hat{\boldsymbol{k}}\cdot\boldsymbol{x}\ , \qquad \hat{\boldsymbol{k}} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}\ . \tag{8.35}$$

Thus, apart from the Laplacian, all sheets contributing to $\psi(\boldsymbol{x},\sigma_3)$ go through the point $\boldsymbol{x}$.

8.7 Information Loss

The reasons why information loss may be an essential ingredient in deterministic hidden variable models of the sort pioneered above, has been extensively discussed in [8,10]. A prototype microcosmos with information loss is shown in Fig. 8.1. Following the arrows, one would conclude that the evolution matrix is

$$U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\ . \tag{8.36}$$

Of course, this is not a unitary matrix. One way to restore unitarity would be to remove state 4. The problem with that is that, in universes with a large number of allowed states, it would be very difficult to determine which of the states are like state number 4, i.e., have no state at all in their (distant) past.

A preferred way to proceed is therefore to introduce equivalence classes of states. Two states are equivalent iff, some time in the near future, they evolve into one and the same state.[1] In Fig. 8.1, states 1 and 4 are equivalent, so they form one class. By construction then, equivalence classes evolve uniquely into equivalence classes.

It should be emphasized that, at the Planck scale, information loss is not a small effect but a very large effect. Large numbers of 'ontological' states are in the same equivalence class, and the equivalence classes form a much smaller set than the class of all states. This is how it can happen that the total number of distinguishable quantum states (i.e., the number of equivalence classes) may only grow exponentially with the surface of a system, whereas the total number of ontological states may rise exponentially with the volume. This seems to be demanded by black hole physics, when we confront the laws of quantum mechanics with those of black holes.

[1] It could also happen that two states merge into the same state in the distant future, but in many models such events become increasingly unlikely as time goes on.

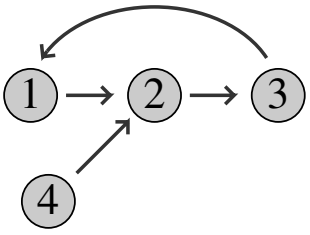

Fig. 8.1. Mini-universe with information loss. *Arrows* show the evolution law

Information loss at the level of the underlying deterministic theory may also explain the apparent lack of causality in the usual attempts to understand quantum mechanics in terms of hidden variables. The definition of an equivalence class refers to the future evolution of a system, and therefore it should not be surprising that in many hidden variable models, causality seems to be violated. One has to check how a system will evolve, which requires advance knowledge of the future.

Information loss at the Planck scale may also shed further light on the origin of gauge theories. It could be that, at the level of the ontological degrees of freedom at the Planck scale, there is no local gauge symmetry at all, but in order to describe a physical state, that is, an equivalence class, we need to describe a particular member of this class, a single state. Its relation to the other members of the same equivalence classes could be what is presently called a 'gauge transformation'.

There is another aspect to be considered in theories with information loss. Theories with continuous degrees of freedom would have an infinite number of possible states if there were no information loss. With information loss, there may be a discrete set of limit cycles, meaning that the equivalence classes may still form discrete sets. Discreteness, one of the prime features of quantum physics, could thus be ascribed to information loss.

8.8 Conclusion

Our view concerning the quantum mechanical nature of our world can be summarized as follows:

- Nature's fundamental laws are defined at the Planck scale. At that scale, all we have is bits of information.
- A large fraction of this information gets lost very quickly, but it is replenished by information entering from the boundaries.
- A quantum state is defined to be an equivalence class of states all of which have the same distant future. This definition is non-local and acausal, which implies that, if we attempt to describe everything that happens

purely in conventional quantum mechanical terms, as in superstring theories, locality and even causality will seem to be absent at the Planck scale. Only in terms of a deterministic theory can this demand of internal logic be met.

- These equivalence classes are described by observables that we call 'beables'. In quantum terminology, beables are a complete set of operators that commute at all times [see (8.24)]. A beable describes what a Planckian observer would be able to register about a system – information that did not get lost.
- All other quantum operators are 'changeables', operators that do not commute with all beables.
- The wave function ψ has the usual Copenhagen–Bohr–Born interpretation.
- However, many or all of the familiar symmetries of nature, such as translation, rotation, Lorentz and isospin symmetry, must be symmetries that relate beables to changeables. This means that the 'ontological' theory behind quantum mechanics does not have these symmetries in a conventional form.

When we go from the Planck scale to the standard model scale:

- Our only way to obtain effective laws of physics at the larger distance scales is by applying the renormalization group procedure.
- Beables and changeables are then mixed up to the extent that it is impossible to identify them; they obey the same laws of physics.
- When we perform a typical quantum experiment, we do not therefore know in advance whether an operator we are working with is a beable or a changeable. Due to the symmetries mentioned above, beables and changeables may obey the same laws of physics. Only when we measure something, and not before, do we know that what we are looking at is a beable. In this way, we believe, apparent clashes with Bell's inequalities may be avoided.
- The classical observables in the classical (macroscopic) limit commute with the beables. They are beables as well.

There remain numerous difficulties. Most urgent is the need for a viable model, demonstrating the workings of the mechanism that we believe to be responsible for the conspicuous quantum mechanical nature of the world we live in. It continues to be difficult to produce a non-trivial model, for instance, one showing particles that interact, while their Hamiltonian is nevertheless bounded from below.

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9 Relational Quantum Mechanics

Carlo Rovelli

QM is one of the most successful scientific theories ever. However, the interpretation of what the theory actually tells us about the physical world has raised a lively debate, which has continued from the early days of the theory to the present.

The possibility that the physical content of an empirically successful physical theory could be debated should come as no surprise: examples abound in the history of science. For instance, the great scientific revolution was fueled by the debate over whether the effectiveness of the Copernican system could be taken as an indication that the Earth was not in fact at the center of the universe. In more recent times, Einstein's celebrated first major theoretical success, special relativity, consisted to a large extent just in understanding the physical meaning (simultaneity is relative) of an already existing effective mathematical formalism (the Lorentz transformations). In these cases, as in the case of quantum mechanics, a very strictly empiricist position could have circumvented the problem altogether, by reducing the content of the theory to a list of predicted numbers. But science would not have progressed in this way.

Quantum theory was first constructed for describing microscopic systems (atoms, electrons, photons) and the way these interact with macroscopic apparatus built to measure their properties. Such interactions are called measurements. The theory is formed by a mathematical formalism, which allows probabilities of alternative outcomes of such measurements to be calculated. If used just for this purpose, the theory raises no difficulty. But we expect the macroscopic apparatus itself – in fact, any physical system in the world – to obey quantum theory, and this seems to raise contradictions in the theory.

9.1 Where is the Problem?

In classical mechanics, a system S is described by a certain number of physical variables. For instance, an electron is described by its position and its spin (intrinsic angular momentum). These variables change with time and represent the contingent properties of the system. We say that their values determine, at every moment, the 'state' of the system. A measurement of a system's variable is an interaction between the system S and an external

system O, whose effect on O depends on the actual value q of the variable (of S) which is measured.

The characteristic feature of quantum mechanics is that it does not allow us to assume that all variables of the system have determined values at every moment (this irrespective of whether or not we know such values). It was Werner Heisenberg who first understood the need to free ourself from the belief that, say, an electron has a well determined position at every time. When it is not interacting with an external system that can detect its position, the electron can be 'spread out' over different positions. In the jargon of the theory, one says that the electron is in a quantum superposition of two (or many) different positions.

It follows that the state of the system cannot be captured by giving the value of its variables. Instead, quantum theory introduces a new notion of the state of a system, which is different from a list of values of its variables. Such a new notion of state was developed in the work of Erwin Schrödinger in the form of the 'wave function' of the system. P.A.M. Dirac gave the general abstract formulation of the notion of quantum state, in terms of a vector Ψ moving in an abstract vector space. From the knowledge of the state Ψ, we compute the probability of the different measurement outcomes q. That is, the probability of the different ways in which the system S can affect a system O in the course of an interaction with it.

The theory then prescribes that at every such measurement, one must update the value of Ψ, to take into account which of the different outcomes has happened. This sudden change of the state Ψ depends on the specific outcome of the measurement and is therefore probabilistic. It is called the collapse of the wave function.

The problem of interpreting quantum mechanics then takes different forms, depending on the relative ontological weight we choose to assign to the wave function Ψ or, respectively, to the sequence of measurement outcomes $q_1, q_2, q_3, \ldots$.

9.1.1 Wave Function Ontology

If we take Ψ as the 'real' entity which fully represents the actual state of affairs of the world, we encounter a number of difficulties. First, we have to understand how can Ψ change suddenly in the course of a measurement: if we describe the evolution of two interacting quantum systems in terms of the Schrödinger equation, no collapse happens. Furthermore, the collapse, seen as a physical process, seems to depend on arbitrary choices in our description and shows a disturbing amount of nonlocality.

But even if we circumvent the collapse problem, the most serious difficulty with this point of view is that it appears to be impossible to understand how specific observed values $q_1, q_2, q_3, \ldots$, could emerge from Ψ alone.

9.1.2 Ontology of Quantum Events

A better alternative is to take the observed values $q_1, q_2, q_3, \ldots$, as the actual elements of reality, and view Ψ as a mere bookkeeping device, determined by the actual values $q_1, q_2, q_3, \ldots$, that happened in past. From this perspective, the real events of the world are the 'realizations' (the 'coming to reality', the 'actualization') of the values $q_1, q_2, q_3, \ldots$, in the course of the interaction between physical systems. These quantum events have an intrinsically discrete (quantized) granular structure.

The difficulty with this second option is that if we take the quantum nature of all physical systems into account, the statement that a certain specific outcome q 'has happened' can be true and not-true at the same time.

To emphasize this key point, consider the case in which a system S interacts with another system (an apparatus) O, and exhibits a value q of one of its variables. Assume that the system O obeys the laws of quantum theory as well, and use the quantum theory of the combined system formed by O and S in order to predict the way this combined system can later interact with a third system O'. Then quantum mechanics forbids us to assume that q has happened. Indeed, as far as its subsequent behavior is concerned, the combined $S + O$ system may very well be in a quantum superposition of different possible values $q_1, q_2, q_3, \ldots$.

This 'second observer' situation captures the core conceptual difficulty of the interpretation of quantum mechanics: reconciling the possibility of quantum superposition with the fact that the observed world is characterized by uniquely determined values $q_1, q_2, q_3, \ldots$. More precisely, it shows that we cannot disentangle the two: according to the theory, an observed quantity q can be at the same time determined and not determined.

9.2 A Solution

A solution to the difficulty is offered by the relational interpretations of quantum mechanics [2]. These are based on the idea that the theory should be understood as an account of the way distinct physical systems affect each other when they interact – and not the way physical systems 'are'. This account exhausts all that can be said about the physical world. The physical world must be described as a net of interacting components, where there is no meaning to 'the state of an isolated system'. The state of a physical system is the net of the relations it entertains with the surrounding systems. The physical structure of the world is identified as this net of relationships.

More precisely, the way out of this dilemma suggested by the relational interpretations is that the values of the variables of a physical system S, namely the qs, are relational. That is, they do not express a property of the system S alone, but rather refer to the relation between two systems.

In particular, the central tenet of relational quantum mechanics [1] is that there is no meaning in saying that a certain variable of the system S takes the value q: rather, there is meaning in saying that this variable takes the value q for O, or with respect to O. The apparent contradiction between the two statements that a variable has or does not have a value is resolved by indexing the statements with the different systems with which the system in question interacts. If I observe an electron at a certain position, I cannot conclude that the electron is there: I can only conclude that the electron as seen by me is there. The unique account of the state of the world of the classical theory is thus fractured into a multiplicity of accounts, one for each possible 'observing' physical system. That is:

> Quantum mechanics is a theory about the physical description of physical systems relative to other systems, and this is a complete description of the world.

9.3 Consistency

This relativisation of actuality is viable thanks to a remarkable property of the formalism of quantum mechanics.

John von Neumann was the first to notice that the formalism of the theory treats the measured system S and the measuring system O differently, but the theory is surprisingly flexible on the choice of where to put the boundary between the two. Different choices give different accounts of the state of the world (for instance, the collapse of the wave function happens at different times); but this does not affect predictions concerning the final observations. This flexibility reflects a general structural property of quantum theory which guarantees the consistency among all the distinct 'accounts of the world' given by the different observing systems. The manner in which this consistency is realized, however, is subtle.

As a simple illustration of this phenomenon, consider the case in which a system O with two states (say, an LED which can be on or off) interacts with a two-state system S (say, the spin of the electron, which can be up or down). Assume the interaction is such that if the spin is up (down) the LED goes on (off). To start with, the electron can be in a superposition of its two states. In the account of the state of the electron that we can associate with the LED, the wave function of the electron collapses to one of two states during the interaction, and the LED is then either on or off. But we can also consider the LED/electron composite system as a quantum system and study the interactions of this composite system with another system O'. In the account associated with O', there is no collapse at the time of the interaction, and the composite system is still in the superposition of the two states (spin up/LED on) and (spin down/LED off) after the interaction. It is necessary to assume this superposition because it accounts for measurable interference

effects between the two states: if quantum mechanics is correct, these interference effects are truly observable by O'. We thus have two discordant accounts of the same events.

But can the two discordant accounts be compared and would such a comparison lead to contradiction? Indeed, they can be compared, because the information on the first account is stored in the state of the LED and O' has access to this information. Therefore O and O' can compare their accounts of the state of the world. However, the comparison does not lead to contradiction because the comparison is itself a physical process that must be understood in the context of quantum mechanics.

Indeed, O' can interact physically with the electron and then with the LED (or, equivalently, the other way around). If, for instance, he finds the spin of the electron up, quantum mechanics predicts that he will then consistently find the LED on, because in the first measurement the state of the composite system collapses to its (spin up/LED on) component.

Hence, the multiplicity of accounts leads to no contradiction precisely because the comparison between different accounts can only be a physical quantum interaction. This internal self-consistency of the quantum formalism is general, and it is perhaps its most remarkable aspect.[1] This self-consistency is taken in relational quantum mechanics as a further indication of the relational nature of the world.

9.4 Correlation

What appears with respect to O as a measurement of the variable q (with a specific outcome), appears with respect to O' simply as a dynamical process that establishes a correlation between S and O. As far as the observer O is concerned, a property q of a system S has taken a certain value. As far as the second observer O' is concerned, the only relevant element of reality is that a correlation is established between S and O.

Concretely, this correlation would appear in all further observations that O' would perform on the $S + O$ system. That is, the way the two systems S and O will interact with O' is characterized by the fact that there is a correlation. In other words, O' will find some properties of O correlated with some properties of S.

[1] In fact, one may conjecture that this peculiar consistency between the observations of different observers is the missing ingredient for a reconstruction theorem of the Hilbert space formalism of quantum theory. Such a reconstruction theorem is still unavailable. On the basis of reasonable physical assumptions, one is able to derive the structure of an orthomodular lattice containing subsets that form Boolean algebras, which 'almost', but not quite, implies the existence of a Hilbert space and its projector algebra. Perhaps an appropriate algebraic formulation of the condition of consistency between subsystems could provide the missing hypothesis to complete the reconstruction theorem.

On the other hand, up to the time at which it interacts physically with $S + O$, the system O' has no access to the actual outcomes of the measurements performed by O on S. This actual outcome is real only with respect to O.

9.5 Information

The existence of a correlation between the possible outcomes of a measurement performed by O' on S and the outcomes of a measurement performed by O' on O can be interpreted in terms of information. In fact, it corresponds precisely to Shannon's definition of information. According to this definition, the statement that O has information about S actually means that we shall observe O and S in a subset of the set formed by the Cartesian product of the possible states of O and the possible states of S. Thus, a measurement of S by O has the effect that O has information about S, in this technical sense, regarding the possible outcomes of the observations by a third system O'.

On the other hand, if we interact a sufficient number of times with a physical system S, we can then predict (the probability distribution of the) future outcomes of our interactions with this system. In this sense, by interacting with S we can say we 'have information' about S. (This information need not be stored or utilized, but its existence is the necessary physical condition for being able to store it or utilize it for predictions.)

Therefore we have two distinct senses in which the physical theory is about information. But a moment of thought shows that the two simply reflect the same physical reality, as it affects two different systems. On the one hand, O has information about S because it has interacted with S and the past interactions are sufficient to 'give information' about future interactions. On the other hand, O has information about S in the sense that there are correlations in the outcomes of measurements that O' can make on the two.

There is a crucial subtle difference, that can be figuratively expressed as follows: O knows about S, while O' only knows that O knows about S, but does not know what O knows. As far as O' is concerned, a physical interaction between S and O establishes a correlation: it does not select an outcome.

But I think that these observations are sufficient to conclude that what precisely quantum mechanics is about is the information that physical systems have about each other. This is the end point of the analysis in [1] and is also the starting point of an attempt at a full reconstruction of the quantum formalism in terms of information theory. I refer the reader to that paper. Concerning this argument, see also [3] and the numerous references therein.

9.6 The Reason for the Apparent Paradoxes

If the suggestions above are correct, the common uneasiness with taking quantum mechanics as a fundamental description of nature (the measurement problem) could then derive from the use of an incorrect notion, rather like the uneasiness with the Lorentz transformations before Einstein derived them from the notion of observer-independent time. The incorrect notion that generates the uneasiness with quantum mechanics is the notion of observer-independent state of a system, or observer-independent values of physical quantities.

All systems can be assumed to be equivalent, there is no observer–observed distinction, and the theory describes only the information that systems have about one another. Nevertheless, the theory is complete.

9.7 Is Quantum Relationalism Connected with Spacetime Relationalism?

The relationalism at the core of quantum theory as pointed out by the relational interpretations may be connected with the spatiotemporal relationalism that characterizes general relativity. Quantum mechanical relationalism is the observation that there are no absolute properties: properties of a system S are relative to another system O with which S is interacting. General relativistic relationalism is the fact that there is no absolute localization in spacetime: localization of an object S in spacetime is only relative to the gravitational field, or to any other object O, with which S is contiguous.

There is a connection between the two, since interaction between S and O implies contiguity and contiguity between S and O can only be checked via some quantum interaction. But because of the difficulty in developing a consistent and conceptually transparent theory of quantum gravity, so far this suggestion has not been developed beyond the stage of a simple intuition.

9.8 Final Comments

In physics, the idea of deepening our insight into the physical world by relativizing notions previously used as absolute has been applied repeatedly and very successfully. Here are a few examples. The notion of the velocity of an object has been understood as meaningless, unless it is indexed with a reference body with respect to which the object is moving. With special relativity, simultaneity of two distant events has been understood as meaningless, unless referred to a specific state of motion of something. (This something is usually called the 'observer' without, of course, any implication that the observer is human or has any other peculiar property besides having a state of

motion. Similarly, the 'observer system' O in quantum mechanics need not be human or have any other property beside the possibility of interacting with the 'observed' system S.) With general relativity, the position of an object in space and time has been understood as meaningless, unless it is referred to the gravitational field, or to another dynamical physical entity.

The move proposed by the relational interpretations of quantum mechanics has strong analogies with these, but is, in a sense, a much longer jump, since all contingent properties of any physical system are taken to be meaningful only as relative to a second physical system. The claim of the relational interpretations is that this is not an arbitrary move. Rather, it is a conclusion which is difficult to escape, following from the observation –explained above in the example of the 'second observer' – that a variable (of a system S) can have a well-determined value q for one observer O and at the same time fail to have a determined value for another observer O'.

This way of thinking about the world certainly has heavy philosophical implications. The claim of the relational interpretations is that it is nature itself that is forcing us to this way of thinking. If we want to understand nature, our task is not to frame nature into our philosophical prejudices, but rather to learn how to adjust our philosophical prejudices to what we learn from nature.

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10 Matrix Models as Non-Local Hidden Variables Theories

Lee Smolin

The ideas that I am going to present here have to do with a certain kind of hidden variables theories. In this sense the theory is in the tradition that Hiley (Chap. 16) and 't Hooft (Chap. **??**) present in this volume. Like them, I do not believe that QM is good enough to extend to a theory of the whole universe, especially with regards to incorporating the dynamics of space and time. If the foundational problems of QM and the problems of space and time are intimately related, it should have something to do with the theory of gravity.

This work has a strange history. When I was first at post doc, I thought I might work on foundations of quantum mechanics, and I did some bad, ugly version of what I am going to explain here. I wrote it up and I sent it to the journal *Physical Review* and I got back a referee report which was from Abner Shimony in which he said: "This is a wonderful paper, it should be published just as soon as the author can clear up a few small points which I am sure he will have no trouble doing." There followed eighteen single-spaced pages of objections and comments. People who know Abner can realize that I never had the courage to send anything back again after that. So there ended my career in foundations of quantum mechanics. It was a preprint in 1983, and I am very happy for this opportunity to return to that idea.

The plan of this article is as follows. First I will motivate roughly why I think that quantum gravity pushes us in the direction of some hidden variables theory. Next I will detail the formulation of a real hidden variables theory, due originally to E. Nelson [1]. Then I will introduce a theory, partly motivated by quantum gravity, and finally I will sketch the steps of the derivation of quantum mechanics from the proposed theory.

10.1 Motivation

Let me begin by pointing out that, in quantum gravity, there has been a lot of progress. We know how to derive the black hole entropy to every epsilon and every dot. There is a set of works which assume conditions that should hold in a black hole horizon or cosmological horizon and derive the space of states, given an explicit dynamical microscopic description of the states and the observables at a horizon. Then, coarse-graining it by the usual methods,

the Bekenstein–Hawking entropy is derived. And that is just one out of many things that have happened. For us, therefore, the interpretational problems are not academic.

For example, I grew up, if I can say so, listening to Jim Hartle talk about the wave function of the universe and all the issues about how to observe it. A few years later H. Kodama [2] wrote down a wave function of the universe (the Chern–Simons state) which is an exact solution of the Wheeler–DeWitt equation. Now this solution is as good as one can possibly imagine a wave function of the universe to be. It is a state in real quantum field theory, and it satisfies the Wheeler–DeWitt equation with full issues of regularization and finiteness and so forth.

There are many others too. For example, Thomas Thiemann and others have written down explicitly infinite dimensional spaces of them. We also have a candidate for the physical inner product that was formulated by Rovelli [3] using a technique known as spin foams. Again, it does not matter what it is – it is computable. There are people like Dan Christensen, John Baez and others who use computers to calculate approximations to these things, for example, for very small universes. They worry about how to do renormalization group calculations and so forth.

So we have pieces of a quantum theory of gravity. For us these are real issues, not only because we are ambitious and we want to solve profound problems, but because in order to have a discussion about the physical validity of a theory, we have to talk about measuring real observables. And that raises all the issues about observables and quantum theory and quantum cosmology that Hartle raises (Chap. 5).

This brings us to the issue of hidden variables, and hence to Bohm and the Hamilton–Jacobi equation. I am going to be adopting a particular point of view about it, originally due to E. Nelson [1]. Now one reason for talking about hidden variables is the issues that Hartle refers to, about how to do quantum theory in a closed system in a cosmological context. Rovelli (Chap. 9) presents another approach called relational quantum theory, and I think it is fair to say that it is motivated by staring at expressions like the equations mentioned above and asking what on earth we can do with them.

Rovelli's relational quantum mechanics is uncannily reminiscent of the ideas put forward by 't Hooft, which indeed arose at about the same time. So I will address here only the slogans and leave it to their articles to explain how this idea really makes sense. The slogans are that there is no quantum state or Hilbert space of the universe but rather that there is a Hilbert space for each division of the world into observer and system. 't Hooft, and at about the same time, Louis Crane, argued that the quantum states should live on the boundaries of systems. Actually, 't Hooft argues that a Hilbert space has something to do with the boundary, and that there is a dimension roughly of the Hilbert space for each Planck area on the boundary up to a constant. So it is for a finite area boundary and it is a finite-dimensional Hilbert space.

That raises the question of how to incorporate the idea that there are many different Hilbert spaces in a quantum theory of spacetime. Is it just any way we want to divide it? Does the dynamics of spacetime tell us how to divide the world up into systems and observers? One proposal of Fotini Markopoulou was that the division can be made by the *causal structure*, which is a characteristic thing in spacetime. A local observer cannot see the whole universe but can only see the information coming from their causal past. Markopoulou proposed that this division (which Rovelli, 't Hooft, Crane and others were also talking about) should involve the causal structure intrinsically. There is, roughly speaking, a Hilbert space for every causal past of the universe. Hartle and Isham's work has been very influential in this respect. Their work on topos theory and quantum theory gives us confidence that there is a mathematical framework to fit these ideas into.

But then there is a problem. In a quantum theory of spacetime, as well as in classical general relativity, everything is dynamical; everything is up for grabs. There is no fixed background structure. Therefore, that particular proposal that the boundaries, the regions we describe quantum-mechanically, have to do with causal structure, runs into a terrible problem: the causal structure is generated dynamically by the evolution of the gravitational field. So it is not something that sits outside the dynamics but rather inside it. Therefore, attempts like those of Hartle and Isham to wed these ideas to a sophisticated mathematical structure run into difficulties. There is a big silence in the literature as to what to do about the fact that, in general relativity, the causal structure is dynamical. Does this mean that the structure of quantum theory, the algebras, observables and the states, are themselves generated dynamically? Now one might want to reply in the affirmative, but for some of us that is a scary thought, although that is one direction to go from here. That is one conundrum that pushes us beyond conventional quantum theory.

Another conundrum is related to quantum gravity. One might like to say that quantum gravity is just like ordinary quantum mechanics. So at some point we are going to have a path and a growth, and we are going to have a thing which is a history, and we are supposed to sum over the histories and express this in many different ways. Now this is the old Feynman idea: we are supposed to sum over the histories in order to get the quantum amplitudes. That is similar to the spin foam approach which Rovelli has worked so much on [3]. Another approach due to Renata Loll and Jan Ambjørn and collaborators is called the dual formulation of spin network evolution, and these things are closely related. People write down explicit proposals for what is a path and what is a growth for a real theory of quantum spacetime. But then the problem is the appearance of i here, because we are summing over amplitudes which are complex numbers. And the thing we are summing over is the causal structures, which now include the causal structures of spacetime. Again, a lot of progress has been made here. One can go to authors like

Mike Reisenberger, John Barett, Louis Crane, Carlo Rovelli and John Baez, and find quite explicit expressions for this, derived from general relativity in different ways. So we have a theory of quantum gravity and now we have to calculate with it.

How are we supposed to sum over all these complex numbers? When we usually face this in ordinary quantum mechanics and quantum field theory, the instruction given is to continue the time variable to i times the time variable. This becomes the exponential of minus a Hamiltonian for a classical statistical mechanics system in one higher dimension. We then just treat it like a statistical mechanics system.

However, the problem is that, because we are summing over the causal structure, time is not necessarily defined for the whole structure. We have to work to define it. Not to mention the fact that this is general relativity, or related to general relativity, and there is no well-defined time on each history. How we do the continuation is an issue. We can make a choice. Renata Loll and Jan Ambjørn do this, in what I think is very important work, because they actually manage to calculate some things. They do it by making on each history a choice of a time variable and by making each continuation with that choice, and they get very interesting and surprising answers. But still, they get something that should worry us because it is not at all obvious that the answers that they got are not fixing some notion of time in a way that relativity might be unhappy with. So that is another conundrum. We seem to be either forced to have preferred notions of time or to face the problem that physicists have never faced, namely, that this is really a sum over complex numbers, and what Feynman wrote down really does not make any sense as a sum over complex numbers.

Now this is the point when somebody usually stands up and says: "Oh well, string theory solves everything." But unfortunately, it does not! In fact, this theory is a strange beast. It is a very interesting research program, full of ideas and ambition, but as a body of results, it is a collection of theories which is opposed to what I have been talking about, because they are background-dependent. That is, each string theory requires a classical background spacetime to define it in the first place. So there are many background-dependent string theories, each defined as an expansion around a classical spacetime background.

Now, there is a conjecture known as the $\mathcal{M}$ theory conjecture which says that there is a real theory behind all of that, in the same sense that if we read Bjorken and Drell [4], we can see that in the 1960s people were speculating that there really was something called quantum field theory behind the perturbation theory of QED. In the second volume [5], they attempted to define it. Similarly, there is a conjecture that there exists a 'second volume' of string theory, analogous to the second volume of Bjorken and Drell. The conjecture says that there is one background-independent theory, to which the classical backgrounds of general relativity of space and time emerge as

classical solutions. Interestingly, at least in several popular formulations of this idea, the degrees of freedom are just a collection of matrices, since there is no spacetime.

I first got involved in this with Richard Levine, who was a student in Imperial College and who is now a student with Rovelli, when we tried to quantize these theories. What we wanted to do, and imagined that we were doing, was to quantize this theory of matrices. What we were actually doing, however, was picking a classical solution to the theory which defined the spacetime background and quantizing with respect to the causal structure and the definition of time, and so forth, on that classical background. And neither we nor anybody else were able to determine how to make a quantum theory related to string theory in which we did what must be the right thing. We did not know whether this ambition of unifying them together was right, i.e., we did not know whether one should quantize the whole theory first and then study different semi-classical solutions to it, or alternatively, have a classical theory, study different classical solutions to it, and define a spacetime on each classical solution.

So that is another reason why quantum theory seems to be failing us. Perhaps all these things in quantum gravity are telling us that QM breaks down. Perhaps at some point we should just stop hitting our head against the wall and imagining that one day somebody will come along and tell us how to solve these problems.

Here I could have mentioned the problems of black holes and the effect of causal structure on quantum theory, as 't Hooft has done (Chap. **??**). And those also weigh very heavily on us. This also does not get us anywhere. It is one thing to observe that quantum mechanics breaks down, and quite another to know what to do about it. The question is: does the way in which it seems to be breaking down give any hints for what lies beyond quantum theory? That is what I am talking about, and I want to explore the point of view that, maybe, the theory which QM breaks down to has something to do with *matrices*. I take as first evidence for this the fact that string theory, loop quantum gravity and spin foams, as well as some work of Rovelli and others, can be expressed in terms where the degrees of freedom are big matrices. And there is a model in which the matrices interact with each other, and indeed, Rovelli et al. showed how we can get quantum spacetime as defined in diagrams in the expansion of the theory of some set of matrices.

Now Anton Zeilinger made the following comment to me: Suppose I have a theory where the degrees of freedom are matrices, and I am supposed to take these matrix models and quantize them. Yet the degrees of freedom are matrices. Now if I quantize that, I am just going to get bigger matrices in which each matrix element of these matrices gets blown up to a bigger matrix. But a matrix of matrices is just a matrix!

So maybe I already have the quantum theory in there, especially since I let these matrices get very big. Maybe, then, the quantum theory of the

matrices is already hidden in the classical theory of the matrices? After all, our attempts are to make the theory invariant under either orthogonal transformation (if they are real) or unitary transformation. That is supposed to be a symmetry of the quantum theory, but the theory is already invariant under those symmetries.

Allow me to give a hand-waving argument why a theory behind quantum theory might be a theory of matrices. As we know, Bell's theorem and the experiments tell us that quantum theory is supposed to be non-local. Rovelli and in a different way 't Hooft tell us that the quantum theory has to do with *relations* between different parts of the universe, maybe crossing a boundary, as in 't Hooft's way of thinking. Similarly, the spacetime of general relativity is about *relationships*. It is not an absolute fixed structure as in the Newtonian space and time. So maybe the fundamental theory is also supposed to be about relationships? And if it is supposed to be about relationships, perhaps the degrees of freedom should be not one, two or three degrees of freedom for each entity, telling us where it is in the background spacetime, which is non-dynamical; instead, maybe there should be degrees of freedom between *pairs of entities*, telling us how they *relate* to each other. And that is how I would argue that the fundamental theory is a theory of matrices. We may think that we have N particles in some D-dimensional space, and talk about some list of their position vectors. But then, maybe, fundamentally, we want to just talk about relationships amongst those entities and construct space itself from a theory of relationships. And that is really where, a long time ago, I started with the theory where the hidden variables were graphs. The fact that quantum theories of gravity, like string theory and spin foams, can be formulated in terms of matrices, seems to be deep, realizing the relational character of space and time. That is what was in the paper that was never published, which I mentioned at the beginning.

So this problem of the background dependence of the quantizations is a major problem for string and $\mathcal{M}$ theory. As the theory classically has many solutions, which define different spacetime backgrounds, it seems that there ought to be a background-independent way to define the quantum theory. If the fundamental theory is a quantum theory the different quantizations defined around the different backgrounds should be approximations to a single exact quantum theory.

This is part of the motivation for the present proposal. Perhaps the background-independent theory is not a conventional quantum theory, but some deeper theory, which can be approximated by conventional quantum theories when the state defines a fixed spacetime background. Such a theory might be approached in different ways, but here we investigate the hypothesis that it might be a hidden variables theory. As we will show, the hint from earlier work is correct, and a matrix model of the kind studied in string and $\mathcal{M}$ theory can serve as a non-local hidden variables theory which can then

reproduce a quantum theory for the eigenvalues of the matrices to leading order in $1/N$.

That is, rather than *quantizing* the *classical* matrix model in some conventional fashion, we will simply assume that the off-diagonal elements of the matrix model are in a classical thermal state. We will find that the quantum theory for the eigenvalues can be reproduced so long as the temperature is scaled in a certain way with N. This formulation of the quantum theory is by definition background-dependent, because the definition of the thermal ensemble makes no reference to any particular classical solution.

To see how this happens it is useful to consider the diagonal elements of the matrices, which become increasingly free at low temperatures, as analogous to the pollen grains in classical Brownian motion. The off-diagonal elements are then analogous to the molecules whose constant collisions with the grains cause them to move with a Brownian motion. Indeed, the diagonal elements are subject to random forces from their interactions with the off-diagonal elements. The off-diagonal elements are small at low temperature, but as we increase N their effects on the diagonal elements are greater due to their greater number. The result is that the interaction of a diagonal elements with a large number of off-diagonal elements introduces a Brownian motion, which is transferred to a Brownian motion of the eigenvalues at low temperature. That is, the randomness of the local variables – the diagonal elements and the eigenvalues – is due to their interactions with a much larger number of non-local variables.

To find interesting behavior, we have to scale T in an appropriate way with N as we take the former to zero and the latter to infinity. In fact, we find that the model behaves critically when we scale the temperature so that $T \approx 1/N$. In this case the off-diagonal matrix elements are of order $1/\sqrt{N}$. However their collective effects on the diagonal elements, and hence the eigenvalues, remain as $N \to \infty$. One such effect is that the diffusion constant which measures the Brownian motion of the eigenvalues remains finite as $N \to \infty$.

Of course, the idea that quantum statistics might just be ordinary statistics in an unusual context is an old one. In particular, Nelson [1] has proposed a *stochastic formulation of quantum theory*, according to which the quantum description of a particle is derived by modifying the classical description solely by the addition of a universal Brownian motion, which satisfies certain special properties. Chief amongst them is that the Brownian motion is non-dissipative, in that energy and momentum are still conserved. Nelson shows in [1] that, when a classical particle is subject to such a non-dissipative Brownian motion, its probability density and current evolve in a way which is equivalent to that given by the Schrödinger equation.

Nelson's formulation plays a key role in the present work, in that we show that the stochastic formulation of quantum theory is recovered to leading order in $1/N$, for the eigenvalues of the matrices in our model. (Note that

one can quantize the free bosonic string directly using Nelson's stochastic quantum theory [6].)

In the present paper we study a bosonic matrix model, and show that it is indeed a non-local hidden variables theory. The extensions of this work to the supersymmetric matrix models associated with string and $\mathcal{M}$ theory are in progress with Stephon Alexander [7].

In the next section we describe a variational principle related to Nelson's formulation of quantum theory and review the basics of the theory of Brownian motion for those unfamiliar with it. In Sect. 10.3, we introduce the matrix model we will study and in Sect. 10.4, we describe the basic physical picture which suggests the connection between the classical statistical mechanics of the matrix model and quantum theory. In Sect. 10.5, we estimate the dependence of the relevant diffusion constants on N, T and other parameters. Finally, the derivation of the Schrödinger equation for the eigenvalues is given in Sect. 10.6.

10.2 Hamilton–Jacobi Theory and Quantum Brownian Motion

To set things up we have to introduce Nelson's work [1], which is not easy for a non-mathematician to understand, so I have rewritten it in terms of Hamilton–Jacobi theory. Although this is partly done in Nelson's own papers, so I do not claim any great originality for this, there is one new idea that I did not see in Nelson's papers, which I call the statistical variational principle. But let us begin with Hamilton–Jacobi theory.

Consider a dynamical system living on an n-dimensional configuration space coordinatized by x^a. The dynamics can be described in terms of a Hamilton–Jacobi function $S(x,t)$, which solves the Hamilton–Jacobi equation

$$\dot{S} + \frac{1}{2m}(\partial_a S)^2 + U = 0 \; . \tag{10.1}$$

A particular solution S defines a family of classical trajectories whose momenta at any point x^a are defined by

$$p_a(x) = \partial_a S \; . \tag{10.2}$$

There are many solutions to the Hamilton–Jacobi equation, each of which defines a congruence of classical trajectories. A statistical description of the system may be given in terms of a probability density $\rho(x,t)$ and a probability current $v^a(x,t)$, which together satisfy

$$\dot{\rho} + \partial_a(\rho v^a) = 0 \; . \tag{10.3}$$

Since the probability is conserved, we may always assume $\int \mathrm{d}^n x \rho(x,t) = 1$.

Now we will do something unusual. Let us restrict attention to an ensemble of classical trajectories whose evolution is determined by a particular solution S of the Hamilton–Jacobi equation. We may call this an S-ensemble. These have a probability density $\rho_S(x,t)$. Since the momentum is determined by S, so must be the probability current. We then have

$$mv_a = \partial_a S \ , \tag{10.4}$$

so that the probability conservation equation is now

$$\dot{\rho} + \frac{1}{m}\partial_a(\rho\partial^a S) = 0 \ . \tag{10.5}$$

The restriction of attention to a statistical ensemble with fixed solution S is unusual, but it does not take us out of the domain of classical physics. The total probability density may be recovered formally as

$$\rho_{\text{total}} = \sum_S \rho_S \ , \tag{10.6}$$

where the sum is over all solutions to the Hamilton–Jacobi function. Of course, the probability current does not add.

We now observe that such ensembles are given by the solutions of a simple variational principle:

$$I[\rho, S] = \int \mathrm{d}t \int \mathrm{d}^n x \rho(x,t) \left[\dot{S} + \frac{1}{2m}(\partial_a S)^2 + U(x)\right] \ . \tag{10.7}$$

The equations that arise from varying ρ and S are, respectively, the Hamilton–Jacobi equation (10.1) and the probability conservation equation (10.5). Thus, the variational principle describes an ensemble of trajectories, each of which evolves according to the Hamilton–Jacobi equation, so that the current velocity is proportional to $\partial_a S$. We note further that the action and equations of motion are invariant under time reversal with

$$t \to -t \ , \quad S \to -S \ , \quad \rho \to \rho \ . \tag{10.8}$$

So here is a variational principle where the variables are a probability distribution and a Hamilton–Jacobi function, and the equations of motion tell us that the Hamilton–Jacobi equation is true, and they tell us that probability is conserved for particles that follow the flow of the Hamilton–Jacobi function. I call it the statistical variational principle. Now various people's ideas, including Bohm's theory, can be expressed by saying that we add a term called the quantum potential in (10.7), and then we get the same conservation law as before, but we get this new quantum Hamilton–Jacobi equation that Hiley refers to (Chap. 16). They follow from the variational principle. Introducing $\hbar$ in an appropriate way, we get the Schrödinger equation [see (10.24) and (10.25) below].

Although this is a well charted ground, I think it is very important and should not be forgotten. The good thing about thinking about quantum physics this way is that conservation of probability is automatic because ρ was always a probability distribution. We automatically have this symmetry that we can add a constant to the Hamilton–Jacobi function and that just changes the phase. But now we have two questions: a big question and an annoying question. The big question is: Where does this quantum potential term come from? The annoying question is: How did S really become a phase? That is, is it really a phase? If we had a configuration space with non-trivial topology, we would really have to worry that it was a phase. And people who want to be annoying ask that question. But I will save it till the end.

Now, Nelson had an answer to the big question. He said that we can forget the question of quantum spacetime! The quantum potential has a natural explanation in terms of the theory of Brownian motion. I think this is a neat idea. To explain it, let us recall that the basic idea of Brownian motion is that we are dealing with a particle which is subject to noise. Therefore trajectories are not smooth. Of course, we really want to think that they are not smooth on some coarse-grained scale but may be smooth on some smaller scale. There are complications here that need careful treatment when we study the theory of Brownian motion. However, from a physical point of view, this just means that we have to distinguish between taking a derivative to the future and taking a derivative to the past.

So Nelson's stochastic version of quantum theory may be formulated in the language we introduced above. To do this we assume that in addition to the classical motion, the particles in our ensemble are subject to a Brownian motion. This Brownian motion is, however, unusual, in that it does not alter the condition that each trajectory in the ensemble is governed by the same solution of the Hamilton–Jacobi equation. We will see that this requirement can be met by making a slight alteration to the Hamilton–Jacobi equation itself, to take into account the fact that trajectories undergoing Brownian motion are not differentiable. Nelson calls the resulting Brownian motion dissipationless Brownian motion, as energy and momentum are still conserved. To describe this dissipationless Brownian motion we may use the language of stochastic differential equations (for reviews see [1]). In this language the small change in time of a trajectory is given by

$$\mathrm{D}x^a = b^a\big(x(t),t\big)\mathrm{d}t + \Delta w^a \,, \qquad \mathrm{d}t > 0 \,, \tag{10.9}$$

for small positive changes in time and

$$\mathrm{D}^* x^a = -b^{*a}\big(x(t),t\big)\mathrm{d}t + \Delta w^{*a} \,, \qquad \mathrm{d}t < 0 \,, \tag{10.10}$$

for small negative changes in time. b^a and b^{*a} are called the forward and backward drift velocities. They describe the average motion of the particles in the ensemble. They are defined by

$$b^a(x,t) = \lim_{\Delta t \to 0} \left\langle \frac{x^a(t+\Delta t) - x^a(t)}{\Delta t} \right\rangle_{x(t)=x} \tag{10.11}$$

and

$$b^{*a}(x,t) = \lim_{\Delta t \to 0} \left\langle \frac{x^a(t) - x^a(t-\Delta t)}{\Delta t} \right\rangle_{x(t)=x} . \tag{10.12}$$

The different elements of the ensemble are distinguished by their Brownian motion, which is given by a Markov process defined by

$$\left\langle \Delta w^a \Delta w^b \right\rangle = -\left\langle \Delta w^{*a} \Delta w^{*b} \right\rangle = \nu \mathrm{d}t q^{ab} \tag{10.13}$$

and

$$\left\langle \Delta w^a \Delta w^{*b} \right\rangle = 0 . \tag{10.14}$$

Here q^{ab} is a metric on the configuration space and ν is the diffusion constant. The averages $\langle \dots \rangle$ are defined with respect to the ensemble. Thus, for any function $F(x)$ on the phase space

$$\langle F \rangle = \int \mathrm{d}^n x \rho(x,t) F(x) . \tag{10.15}$$

From these basic definitions, one can easily derive the forward and backward Fokker–Planck equations

$$\dot{\rho} = -\partial_a(\rho b^a) + \nu \nabla^2 \rho , \qquad \dot{\rho} = -\partial_a(\rho b^{*a}) - \nu \nabla^2 \rho . \tag{10.16}$$

The current conservation equation follows from these with

$$v^a = \frac{1}{2}(b^a + b^{*a}) . \tag{10.17}$$

The difference between the forward and backward drift velocities is called the osmotic velocity. From the Fokker–Planck equation, it satisfies

$$u^a = \frac{1}{2}(b^a - b^{*a}) = \nu \partial^a \ln \rho . \tag{10.18}$$

We thus see that the diffusion constant measures the extent to which the paths are non-differentiable, so that the forward and backward drift velocities are not equal. This is of course possible because they are defined in (10.11) and (10.12) in such a way that the limit $\Delta t \to 0$ is taken *after* averaging over the ensemble.

Hence, the key thing is that we average over the ensemble before we take the limit $\Delta t \to 0$. And the key point about Brownian motions is that they are not equal because of the noise. So we can average the forward and the backward velocity, and that gives us the current velocity, which is what

comes into the conservation law for probability. But we can also take the difference and show that the difference between them, the osmotic velocity, is always proportional to the gradient of the probability distribution, plus a constant which is called the diffusion constant. Hence, this is the key thing: the diffusion constant measures how much averaging into the future and averaging into the past are different in the theory of Brownian motion.

Now what Nelson noticed is that, if we take $\hbar/2m$ for the diffusion constant in (10.13), then the sum of the ordinary kinetic energy, $mv^2/2$, plus the quantum potential, is just the current velocity squared plus the osmotic velocity squared, times the mass over two. So the key idea is that we can unify the quantum potential energy with the ordinary kinetic energy by thinking about Brownian motion and by hypothesizing that, underlying all that, we are dealing with a particle which has an irreducible Brownian motion on the configuration space. And if we worry about non-locality, etc., the whole point is that the motion is Brownian and on the configuration space. So Nelson says that the transition from classical to quantum realm is due only to the presence of this noise.

Let us see how to formulate this. We want to preserve the property (10.4) that the current velocity is proportional to the gradient of the Hamilton–Jacobi function, but we want to find a way to include Brownian motion within the ensemble defined by a particular Hamilton–Jacobi function. One way to approach this is to modify the statistical variational principle to include the effects of Brownian motion. It is not hard to see that this is possible, and that the right extension of the variational principle is

$$I^\nu[\rho, S] = \int \mathrm{d}t \int \mathrm{d}^n x \rho(x,t) \left[\dot{S} + \frac{1}{2m}(\partial_a S)^2 + \frac{m\nu^2}{2}(\partial_a \ln \rho)^2 + U(x) \right] . \tag{10.19}$$

To see why let us use the fact that $\partial_a S$ is proportional to the momentum. Thus we have for smooth motion, on solutions to the variational principle,

$$\begin{aligned} \int \mathrm{d}t \int \mathrm{d}^n x \rho(x,t) \frac{1}{2m}(\partial_a S)^2 & \\ &= \int \mathrm{d}t \int \mathrm{d}^n x \rho(x,t) \frac{1}{2m}(p^a)^2 \\ &= \int \mathrm{d}t \int \mathrm{d}^n x \rho(x,t) \frac{m}{2}(\dot{x}^a)^2 \\ &= \int \mathrm{d}t \int \mathrm{d}^n x \rho(x,t) \frac{m}{2} \lim_{\Delta t \to 0} \left[\frac{x^a(t+\Delta t) - x^a(t)}{\Delta t} \right]^2 . \end{aligned}$$

However, for a Brownian motion process, the limit in the last line is not defined. So this is not a consistent variational principle when $\nu \neq 0$. To define a variational principle that is well defined for the case of Brownian motion where the paths are non-differentiable we need to take the limit that

defines the derivative outside of the ensemble average. Thus, we may define instead

$$\int \mathrm{d}t \int \mathrm{d}^n x \rho(x,t) \frac{1}{2m} (p^a)^2 \equiv \int \mathrm{d}t \lim_{\Delta t \to 0} \int \mathrm{d}^n x \rho(x,t) \frac{m}{2} \left[\frac{x^a(t+\Delta t) - x^a(t)}{\Delta t} \right]^2 .$$

This form of the integrand appears, however, to lack invariance under time reversal (10.8). This is because, when the paths are non-differentiable, b^a and b^{*a} may not be equal. However, we notice that under the time integral we can write

$$\begin{aligned} &\int \mathrm{d}t \lim_{\Delta t \to 0} \int \mathrm{d}^n x \rho(x,t) \frac{m}{2} \left[\frac{x^a(t+\Delta t) - x^a(t)}{\Delta t} \right]^2 \\ &\quad = \int \mathrm{d}t \lim_{\Delta t \to 0} \int \mathrm{d}^n x \frac{m}{4} \left\{ \rho(x,t) \left[\frac{x^a(t+\Delta t) - x^a(t)}{\Delta t} \right]^2 \right. \\ &\qquad \left. + \rho(x, t-\Delta t) \left[\frac{x^a(t) - x^a(t-\Delta t)}{\Delta t} \right]^2 \right\} . \end{aligned}$$

Now we observe that

$$\rho(x, t-\Delta t) = \rho(x,t) - \Delta t \dot{\rho}(x,t) . \tag{10.20}$$

As $\dot{\rho}(x,t)$ is given by the Fokker–Planck equation, the second term leads to terms that are well behaved and vanish as $\Delta t \to 0$. Thus, we can take the average and then the limit, using (10.11) and (10.12) to find

$$\begin{aligned} &\int \mathrm{d}t \lim_{\Delta t \to 0} \int \mathrm{d}^n x \rho(x,t) \frac{m}{2} \left[\frac{x^a(t+\Delta t) - x^a(t)}{\Delta t} \right]^2 \\ &\qquad = \int \mathrm{d}t \int \mathrm{d}^n x \rho(x,t) \frac{m}{4} \left[(b^a)^2 + (b^{*a})^2 + C \right] \\ &\qquad = \int \mathrm{d}t \int \mathrm{d}^n x \rho(x,t) \frac{m}{2} \left[(v^a)^2 + (u^a)^2 + C \right] . \end{aligned}$$

Here C is an infinite constant, which is equal to

$$C = \nu d \lim_{\Delta t \to 0} \frac{1}{\Delta t} . \tag{10.21}$$

Thus, we have found that we can extract an infinite constant from the action, leaving us with a finite piece that is well defined and time-reversal invariant. So we have,

$$\int \mathrm{d}t \lim_{\Delta t \to 0} \int \mathrm{d}^n x \rho(x,t) \frac{m}{2} \left[\frac{x^a(t+\Delta t) - x^a(t)}{\Delta t} \right]^2$$
$$= \int \mathrm{d}t \int \mathrm{d}^n x \rho(x,t) \left[\frac{1}{2m} (\partial_a S)^2 + \frac{m\nu^2}{2} (\partial_a \ln \rho)^2 \right] + \frac{mC}{2} \ .$$

Constants, even infinite constants, play no role in classical action principles. Hence, putting this last result back in the definition of the action principle, we see that the effect of exchanging the order of the integral and the limit is the new action principle (10.19). Thus we see that while we can have a Brownian motion within the trajectories defined by a given Hamilton–Jacobi functional, the definition of Brownian motion requires that we modify the Hamilton–Jacobi equation, so that we have a consistent variational principle, even in the presence of non-differentiable paths.

From the argument just given we see that the variational principle (10.19) is equivalent to defining the time derivatives so that they are taken outside the ensemble averages. This yields the standard variational principle when the trajectories are smooth, because then it does not matter in which order we take the limits involved in the definition of the time derivatives and the ensemble average. But for non-differentiable paths the order matters and we see that we must take the limit defining the time derivative after that defining the ensemble average. As we have seen, up to an infinite constant which may be ignored, this is equivalent to adding the new term

$$\frac{m\nu^2}{2} (\partial_a \ln \rho)^2$$

to the variational principle.

The new Hamilton–Jacobi equation follows from (10.19) by varying by ρ. We find that

$$\dot{S} + \frac{1}{2m} (\partial_a S)^2 - 2m\nu^2 \frac{1}{\sqrt{\rho}} \nabla^2 \sqrt{\rho} + U = 0 \ . \tag{10.22}$$

The Hamilton–Jacobi functional is thus modified by a new potential term which is a function of the probability density. We recall that this unusual feature follows because it is the gradient of the probability density that measures the importance of the non-differentiability of the paths. The probability conservation, however, is not modified. Now, the big surprise is that (10.22) and (10.5) are nothing else than the real and imaginary parts of the Schrödinger equation, with

$$\Psi(x,t) = \sqrt{\rho(x,t)} \mathrm{e}^{\mathrm{i}S/\hbar} \tag{10.23}$$

and with

$$\hbar = 2\nu m \ . \tag{10.24}$$

Thus, the variational principle (10.19) in the presence of Brownian motion is equivalent to

$$\mathrm{i}\hbar\frac{\mathrm{d}\Psi(x,t)}{\mathrm{d}t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + U(x)\right]\Psi(x,t)\ . \tag{10.25}$$

One way to understand this is the following. From the point of view presented here, a quantum state is nothing more nor less than an ordinary statistical ensemble of Brownian motion trajectories, which share a single Hamilton–Jacobi function S, where that S is itself a solution to a modified Hamilton–Jacobi equation, modified to take into account the change in the definition of the momentum necessary when the motion is Brownian. This is the basic message of Nelson's stochastic quantum theory.

We can find a Hamiltonian formulation of the statistical variational principle. To this end note that (10.19) can be written

$$I^\nu[\rho,S] = \int \mathrm{d}t \int \mathrm{d}^n x\Big(S\dot{\rho} - \mathcal{H}[\rho,S,x]\Big)\ , \tag{10.26}$$

where the Hamiltonian density is

$$\mathcal{H}[\rho,S,x] = \rho\left[\frac{1}{2m}(\partial_a S)^2 + \frac{m\nu^2}{2}(\partial_a \ln\rho)^2 + U(x)\right]\ . \tag{10.27}$$

Thus we see that the probability density ρ can be considered as a conjugate coordinate with S its conjugate momentum, so that we have an infinite dimensional phase space with

$$\{\rho(x), S(x')\} = \delta^n(x',x)\ . \tag{10.28}$$

The Hamiltonian

$$H = \int \mathrm{d}^n x\mathcal{H} \tag{10.29}$$

is then conserved in time. It is easy to check that Hamilton's equations of motion are the Hamilton–Jacobi equation (10.22) and the probability conservation equation (10.5). We note that this is true for any value of the diffusion constant ν so that this is true in both classical and quantum theory. To get more insight into how the linearity of quantum theory has emerged from the theory of Brownian motion, we can write out the conserved Hamiltonian:

$$H = \int \mathrm{d}^n x\rho\left[\frac{1}{2m}(\partial_a S)^2 + \frac{m\nu^2}{2}(\partial_a \ln\rho)^2 + U(x)\right]\ . \tag{10.30}$$

This seems a very non-linear expression, and yet it is nothing but the expectation value of a linear operator. To see this we rewrite it slightly as

$$H = \int \mathrm{d}^n x \sqrt{\rho} \mathrm{e}^{-\mathrm{i}S/\hbar} \left[\frac{1}{2m} (\partial_a S)^2 - 2m\nu^2 \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} + U(x) \right] \sqrt{\rho} \mathrm{e}^{\mathrm{i}S/\hbar} \ . \quad (10.31)$$

Using (10.23) and (10.24), this is easily seen to be equal to

$$H = \int \mathrm{d}^n x \bar{\Psi} \hat{\mathcal{H}} \Psi \ , \quad (10.32)$$

with

$$\hat{\mathcal{H}} = -\frac{\hbar^2}{2m} \nabla^2 + U(x) \ . \quad (10.33)$$

Thus, the very non-linear seeming equation $\dot{H} = 0$ is seen to be actually equivalent to the linear Schrödinger equation. We also see that the conserved Hamiltonian which arises from the statistical variational principle is exactly equal to the expectation value of the Hamiltonian operator in the quantum theory. Thus, the conservation of the Hamiltonian in the statistical variational principle is equivalent to the conservation of the expectation value of the Hamiltonian operator in the quantum theory.

Now the aim in what follows is to recover Nelson's idea from a matrix model of the hidden variables theory. Here lies the novelty. The question is this: If Nelson is right, we are supposed to explain the source of this noise. Where does the noise come from? That is what this theory is about.

10.3 Matrix Model

We study a bosonic matrix model which is the bosonic part of the models used in string and $\mathcal{M}$ theory [8–11]. The degrees of freedom are d $N \times N$ real symmetric matrices $X_{ai}^{\ j}$, with $a = 1, \ldots, d$ and $i, j = 1, \ldots, N$. The action is

$$S = \mu \int \mathrm{d}t \mathrm{Tr} \Big\{ \dot{X}_a^2 + \omega^2 [X_a, X_b][X^a, X^b] \Big\} \ . \quad (10.34)$$

We choose the matrices X^a to be dimensionless. ω is a frequency and μ has dimensions of mass $\times$ length^2. We do not assume $\hbar = 1$. In fact, as we aim to derive quantum mechanics from a more fundamental theory, $\hbar$ is not yet meaningful. We will introduce $\hbar$ as a function of the parameters of the theory when we derive the Schrödinger equation as an approximate evolution law. We may note that the parameters of the theory define an energy $\epsilon = \mu\omega^2$.

The basic idea is that the off-diagonal matrix elements of X^a will be the non-local hidden variables. The physical observables will be defined to be the eigenvalues λ_i^a of the matrices. We will put the system at a small, but finite temperature, the result of which will be that the matrix elements undergo Brownian motion as they oscillate in the potential. It follows from

linear algebra that the eigenvalues also undergo Brownian motion. We will see that the parameters of the theory can be scaled with N in such a way that Nelson's stochastic formulation of quantum mechanics is realized for the eigenvalues. When $T = 0$, the matrices must commute with each other so as to achieve the vanishing of the potential energy. This means that it is possible to simultaneously diagonalize them. When T is finite, but small compared to ϵ, the off-diagonal elements will on average be small. As a result, it is useful to split the matrices into diagonal and off-diagonal pieces:

$$X_{ai}^{\ j} = D_{ai}^{\ j} + Q_{ai}^{\ j} \ , \tag{10.35}$$

where $D^a = \text{diagonal}(d_1^a, \ldots, d_N^a)$ is diagonal and $Q_{ai}^{\ j}$ has no diagonal elements. Since the $Q_{ai}^{\ j}$ are dimensionless, we will expect them to scale as a power of $T/\mu\omega^2$. We then write the action (10.34) as

$$S = \int \mathrm{d}t \left(\mathcal{L}^d + \mathcal{L}^Q + \mathcal{L}^{\text{int}} \right) \ . \tag{10.36}$$

The theory of the d's alone is free,

$$\mathcal{L}^d = \mu \sum_{ai} (\dot{d}_i^a)^2 \ , \tag{10.37}$$

while the theory of the Q's alone has the same quartic interaction,

$$\mathcal{L}^Q = \mu \left\{ \sum_{aij} (\dot{Q}_{ai}^{\ j})^2 + \omega^2 [Q_a, Q_b][Q^a, Q^b] \right\} \ . \tag{10.38}$$

The interaction terms between the diagonal and off-diagonal elements are

$$\mathcal{L}^{\text{int}} = 2\mu\omega^2 \sum_{abij} \Big\{ -(d_i^a - d_j^a)^2 (Q_{bi}^{\ j})^2 - (d_i^a - d_j^a)(d_i^b - d_j^b) Q_{ai}^{\ j} Q_{bj}^{\ i} \tag{10.39}$$

$$+2(d_i^a - d_j^a) Q_i^{bj} [Q_a, Q_b]_j^{\ i} \Big\} \ .$$

We note that the model has a translation symmetry given by

$$d_i^a \longrightarrow d_i^a + v^a \ . \tag{10.40}$$

The result is that the center of mass momentum of the system is conserved.

10.4 The Physical Picture

In the following we will show that the ordinary statistical physics of the model in the last section has a critical behavior when the off-diagonal sector is heated

to finite temperature and the large N limit is taken with the temperature scaled so that $T \approx 1/N$. We will further see that a feature of the critical behavior in this limit is to reproduce quantum mechanics. That is, to leading order in $1/N$ the evolution of the probability density and current for the eigenvalues of the matrixes is equivalent to that given by the free Schrödinger equation.

To show this we will apply what we have learned in Sect. 10.2 to the matrix model. We will formulate an S-ensemble for the matrix model in terms of the statistical variational principle. At the fundamental level the dynamics is formulated in terms of the variational principle without Brownian motion, i.e., the simple variational principle (10.7). Thus, the whole matrix model is in an ordinary statistical ensemble. In the next section we will study the behavior of the matrix model at low temperature and large N. We see that when we pick $T \approx 1/N$, the off-diagonal elements scale as $1/\sqrt{N}$. This makes sense as they must go to zero at $T = 0$. We also see that to leading order the off-diagonal elements move harmonically in an average field given by the average values of all the other off-diagonal elements. The diagonal elements are not required to vanish as $T \to 0$, so they remain of order unity. However, the diagonal elements move in a random potential given by the oscillations of all the off-diagonal elements. The result is that the diagonal elements pick up a random Brownian motion on top of their free motion. This Brownian motion is then also experienced by the eigenvalues. We will see that when the model is scaled critically, the diffusion constants for the diagonal elements and eigenvalues go to constant limits as $N \to \infty$ and $T \to 0$.

We then want to study the effect of the Brownian motion of the eigenvalues. To do this we derive an effective statistical action for the probability distribution of the eigenvalues by averaging the statistical variational principle for the whole model over the values of the matrix elements. This is the task of Sect. 10.6. We see that the Brownian motion term in (10.19) emerges naturally as a term in the effective statistical action for the eigenvalues, as a result of the induced Brownian motion just described. Furthermore, in that limit the conserved energy of the variational principle of the whole model reduces to the conservation of the expectation value of the free Hamiltonian operator for the eigenvalues. Thus, in the large N limit, Nelson's stochastic formulation of quantum theory emerges naturally as a description of the statistical behavior of the eigenvalues.

10.5 Estimates for the Diffusion Constants at Low Temperature

In this section we investigate the consequences of putting the matrix model in a thermal bath at a temperature T. We have two tasks. The first is to understand how various quantities of interest scale with T and N, at low temperatures. The second is to derive estimates for the diffusion constants

for the matrix elements and eigenvalues that are good for low temperature and large N. By low temperature T, we will mean that the ratio $T/\mu\omega^2$ is small. It will be convenient to scale this ratio with N, so we define

$$\frac{T}{8(d-1)\mu\omega^2} = \frac{t}{N^p} \ , \tag{10.41}$$

with t dimensionless and p a power. The factor of $8(d-1)$ is inserted for later convenience. We begin by recalling how the potential is written in terms of diagonal and off-diagonal elements [see (10.39)].

$$\begin{aligned} U(d,Q) = \mathcal{L}_{\rm int} = \mu\omega^2 {\rm Tr}\Big\{ & -2(d_a^i - d_a^j)(d^{ai} - d^{aj})Q_{bij}Q^{bij} \\ & + 2(d_a^i - d_a^j)(d^{bi} - d^{bj})Q_{bij}Q^{aji} \\ & + 4(d_a^i - d_a^j)Q_{bij}[Q^a, Q^b]^{ji} + [Q^a, Q^b]^2\Big\} \ . \end{aligned}$$

The classical equations of motion are then

$$\begin{aligned} \ddot{d}_a^i = \omega^2\Big\{ & -8(d_a^i - d_a^j)Q_{bij}Q^{bij} + 8(d^{bi} - d^{bj})Q_{bij}Q^{aji} \\ & +8Q_{bij}[Q^a, Q^b]^{ji} + 4\big[Q_b, [Q^a, Q^b]\big]_{ij}\Big\} \end{aligned} \tag{10.42}$$

and

$$\begin{aligned} \ddot{Q}_{ij}^a = \omega^2\Big\{ & -4\big[\delta_b^a(d_a^i - d_a^j)^2 - (d_a^i - d_a^j)(d^{bi} - d^{bj})\big]Q_{ij}^b \\ & +4(d_a^i - d_a^j)Q_{bij}^2 - 8(d_b^i - d_b^j)[Q_a, Q_b]_{ij}\Big\} \ . \end{aligned} \tag{10.43}$$

Now we will consider how each matrix element moves in an effective potential created by the averaged motions of the other elements. To see this we make a mean field approximation, good at large N. We assume that the statistical averages satisfy relations consistent with the symmetry of the theory. This gives us

$$\langle Q_a^{ij} Q_b^{kl}\rangle = q^2\delta_{ab}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \ , \tag{10.44}$$

$$\langle (d_a^i - d_a^j)(d_b^k - d_b^l)\rangle = r^2\delta_{ab}(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}) \ , \tag{10.45}$$

$$\langle Q_{aij}\rangle = \langle d_a^i\rangle = \langle dQ\rangle = \langle Q^3\rangle = \langle d^3\rangle = 0 \ . \tag{10.46}$$

We assume also that averages of four matrix elements factor into pairs of averages of two in all ways. Our goal will be to solve for the value of q when the off-diagonal elements are in a thermal bath. We now write out the effective

potential for the matrix elements moving in the averaged fields of the other elements, to quadratic order. This gives us,

$$\langle U \rangle = \frac{\mu \Omega_Q^2}{2} Q_{aij} Q^{aij} + \frac{\mu \Omega_d^2}{2} (d_a^i - d_a^j)^2 \ , \tag{10.47}$$

with

$$\Omega_Q^2 = 4(d-1)\omega^2[(N-1)q^2 + 2r^2] \ , \qquad \Omega_d^2 = 4(d-1)\omega^2 q^2 \ . \tag{10.48}$$

Thus, each off-diagonal element moves in a harmonic potential created by the averaged values of the other elements. The diagonal elements are, to leading order, a system of points, each connected to all the others by springs with the same spring constant. We will make no assumption about the statistical distribution of the diagonal elements. We will see that this is consistent as long as p is chosen so that the diagonal spring constant Ω_d vanishes as $N \to \infty$. We will assume that the Q_{aij} are in thermal equilibrium with each other. This implies that, at temperature T,

$$\frac{\mu}{2} q^2 \Omega_Q^2 = \frac{T}{4} \ , \tag{10.49}$$

which tells us that

$$\frac{T}{8(d-1)\mu\omega^2} = (N-1)q^4 + 2r^2 q^2 \ . \tag{10.50}$$

Now let us assume that we can neglect the term in $r^2 q^2$ for estimating the order of magnitudes of quantities and the dependence on N. This will be consistent so long as $r^2 \leq N q^2$. We then arrive at the estimate that, for large N,

$$q = \frac{1}{N^{1/4}} \left[\frac{T}{8(d-1)\mu\omega^2} \right]^{1/4} = t^{1/4} \frac{1}{N^{(p+1)/4}} \ . \tag{10.51}$$

We see that, for $p = 1$, it is true that Nq^2 is of order unity, so this is consistent also with r being of order unity as $N \to \infty$ and $T \to 0$. Note that we now have

$$\Omega_Q^2 = 4\omega^2(d-1)[2r^2 + \sqrt{t} N^{(1-p)/2}] \ . \tag{10.52}$$

Hence, for Ω_Q not to diverge as $N \to \infty$, we must have $p \geq 1$. We then see that $p = 1$ corresponds to a critical point at which Ω_Q is fixed as $N \to \infty$. We also see from (10.51) that this corresponds to $q \approx 1/\sqrt{N}$ and

$$\frac{\Omega_d^2}{\Omega_Q^2} = \frac{q^2}{Nq^2 + 2r^2} = \frac{1}{N + 2r^2/q^2} \ . \tag{10.53}$$

Therefore, if we choose $p = 1$ so that Ω_Q is fixed in the limit $N \to \infty$, then in the same limit Ω_d vanishes, so the diagonal elements remain free in the mean field approximation. From now on, unless otherwise specified, we will take $p = 1$.

Now we would like to estimate the effects on the matrix elements of the fluctuations around the thermal averages for small temperature and large N. Let us consider for example the contribution of the last term in (10.42) to the motion of d_a^i. We have, neglecting the other terms,

$$\ddot{d}_a^i = 4\omega^2 \sum_{bj} \left[Q_b, [Q^a, Q^b]\right]_{ij} . \tag{10.54}$$

Counting the commutators, this is a sum of dN^3 terms, for each a and i. These add with random signs. Each impulse of a fixed sign lasts an average time of Ω_Q^{-1}, as this is the time over which the values of the Q_{aij} oscillate. The result is that d_a^i has a random Brownian motion on top of the free motion given by the $N \to \infty$ limit. To estimate the diffusion constant that results note that if, in a time Δt, the average displacement resulting from the random forces is Δd, the diffusion constant is

$$\nu_d = \frac{(\Delta d)^2}{\Delta t} . \tag{10.55}$$

If a_{total} is the total acceleration given by the sum of the random forces over the time Δt, we have $\Delta d = a_{\text{total}}(\Delta t)^2/2$. Thus, we have

$$\nu_d = \frac{1}{4} a_{\text{total}}^2 (\Delta t)^3 . \tag{10.56}$$

For a_{total} we may take

$$a_{\text{total}} = \omega^2 q^3 \sqrt{N^3 d} , \tag{10.57}$$

because we are adding $N^3 d$ terms with random signs and average magnitude $\omega^2 q^3$. Taking $\Delta t = \Omega_Q^{-1}$, we have, for large N,

$$\nu_d = \frac{\omega d}{4(d-1)^{3/2}} \frac{q^3 N^{3/2}}{\left(1 + \dfrac{2r^2}{Nq^2}\right)^{3/2}} = \frac{\omega t^{3/4} d}{4(d-1)^{3/2} N^{3(p-1)/4}} \frac{1}{\left(1 + \dfrac{2r^2}{Nq^2}\right)^{3/2}} . \tag{10.58}$$

So we see that if we pick $p = 1$, and so long as $2r^2$ is of order one, the diffusion constant for the diagonal elements goes to a limit which is N-independent and hence T-independent as $N \to \infty$ and $T \to 0$. Under these same conditions we can show that as $N \to \infty$ these terms make the dominant contribution to the random forces on the diagonal elements coming from (10.42). A similar

analysis starting from (10.43) allows us to estimate the diffusion constant for the Q's coming from the random forces to be

$$\nu_Q = \omega \frac{r^2 t}{N^{7p/4+5/4}} \ . \tag{10.59}$$

The result is that for $p \geq 1$, ν_Q vanishes in the limit $N \to \infty$. Now a classic result of random matrix theory is that if the matrix elements of a matrix undergo Brownian motion, so do its eigenvalues. In a case like ours it is clear, as the eigenvalues will be close to the diagonal values, that the off-diagonal values vanish as $N \to \infty$. But we have to be careful about the contributions from higher order terms in perturbation theory. By making use of the standard formula

$$\lambda_i^a = d_I^a + \sum_j \frac{Q_{ij}^a Q_{ji}^a}{d_i^a - d_j^a} + \cdots \ , \tag{10.60}$$

we can show that the diffusion constant for the eigenvalues is given by

$$\nu_\lambda = \nu_d + \frac{N\nu_Q q^2}{r^2} + \cdots \approx \omega \left[\frac{t^{3/2} d}{4(d-1)^{3/2} N^{3(p-1)/4}} + \frac{t^{3/2}}{N^{9p/4+3/4}} \right] \ . \tag{10.61}$$

This tends to ν_d as $N \to \infty$. Thus, we see that the dominant contribution to the Brownian motion of the eigenvalues comes from the Brownian motion of the diagonal elements. These in turn are fluctuating because they are perturbed by their interactions with the off-diagonal elements, which are moving in a harmonic potential, created by their averaged values at finite temperature. The result is that a randomness is introduced into the motions of the eigenvalues, coming from the interactions of the diagonal elements with a very large number of random variables, which are the off-diagonal elements.

This then illustrates that idea that a local degree of freedom can have its motion randomized by interaction with a large number of non-local degrees of freedom. In the next section we will see that this may result in behavior that for large N is indistinguishable from that predicted by the Schrödinger equation.

10.6 Derivation of the Schrödinger Equation

We are now ready to derive the Schrödinger equation for the eigenvalues of the matrices. As we described above this is a three-step process:

- **Step 1.** Formulate the statistical variational principle for the matrix model.
- **Step 2.** Make assumptions about the statistical ensemble. In particular, we assume that the model is in an S-ensemble, heat it to finite temperature T and then study the large N limit with $T \approx 1/N$.

- **Step 3.** Derive an effective statistical variational principle for the eigenvalues by averaging over the variational principle of the matrix elements and show that, when $N \to \infty$, this is equivalent to the Schrödinger quantum theory for the eigenvalues.

Step 1: Statistical Variational Principle for the Matrix Model

We begin by defining an S-ensemble for the matrix elements. That is, we begin with the variational principle

$$I[\rho, S] = \int \mathrm{d}t \int \mathrm{d}d\, \mathrm{d}Q \rho(d, Q, t) \tag{10.62}$$
$$\left\{ \dot{S}(d, Q) + \frac{1}{2\mu} \left[\frac{\delta S(d, Q, t)}{\delta d_i^a} \right]^2 + \frac{1}{2\mu} \left[\frac{\delta S(d, Q, t)}{\delta Q_{ij}^a} \right]^2 + U(d, Q) \right\} ,$$

where $U(d, Q)$ is the interaction term $\mathcal{L}^{\mathrm{int}}$ given by (10.39).

Step 2: Physical Assumptions

We now state the physical assumptions we make concerning ρ and S. These are assumed only to hold to leading order in $1/N$:

- The Q system is in a distribution that is, to leading order in $1/N$, statistically independent of the distribution of the eigenvalues. This means that to leading order the probability density factorizes:

$$\rho(d, Q) = \rho_d(d)\rho_Q(Q) + O(1/N) . \tag{10.63}$$

- The Q subsystem is in thermal equilibrium at a temperature T. So we have

$$\rho_Q(Q) = \frac{1}{Z} \mathrm{e}^{-H(Q)/T} , \tag{10.64}$$

where $H(Q)$ is the Hamiltonian corresponding to the Q system alone, viz.,

$$H(Q) = \mu \left\{ \sum_{aij} (\dot{Q}_{ai}^{\ j})^2 - \omega^2 [Q_a, Q_b][Q^a, Q^b] \right\} , \tag{10.65}$$

and

$$Z = \int \mathrm{d}Q \mathrm{e}^{-H(Q)/T} . \tag{10.66}$$

As a result of these assumptions, our variational principle reads

$$I[\rho_d, S, T] = \int \mathrm{d}t \int \mathrm{d}d\, \mathrm{d}Q \rho_d(d) \rho_Q(Q) \left[\dot{S}(d,Q) + \frac{1}{2\mu} \left(\frac{\delta S}{\delta d_i^a} \right)^2 + \frac{1}{2\mu} \left(\frac{\delta S}{\delta Q_{ij}^a} \right)^2 + U(d,Q) \right] . \tag{10.67}$$

Step 3: Derive an Effective Variational Principle for the Eigenvalues

Now we want to derive an effective variational principle to describe the evolution of the probability distribution for the eigenvalues. We will do this by averaging the variational principle (10.67) over the values of the matrix elements, and then extracting the leading behavior for large N and small T. We begin by inserting the factor unity in the form

$$1 = \int \prod_{ai} \mathrm{d}\lambda_i^a \delta \left(\lambda_i^a - d_i^a - \sum_j \frac{Q_{ij}^a Q_{ji}^a}{d_i^a - d_j^a} + \cdots \right) . \tag{10.68}$$

Thus, we have

$$I[\rho_d, S, T] = \frac{1}{Z} \int \mathrm{d}t \int \mathrm{d}d\, \mathrm{d}Q \int \mathrm{d}\lambda \delta \left(\lambda_i^a - d_i^a - \sum_j \frac{Q_{ij}^a Q_{ji}^a}{\lambda_i^a - \lambda_j^a} + \cdots \right) \rho_d \mathrm{e}^{-H(Q)/T} \left[\dot{S} + \frac{1}{2\mu} \left(\frac{\delta S}{\delta d_i^a} \right)^2 + \frac{1}{2\mu} \left(\frac{\delta S}{\delta Q_{ij}^a} \right)^2 + U(d,Q) \right] .$$

Now we would like to integrate over the d's, which will express the action in terms of only the λ's and Q's. However, before doing this we need to take into account that, to the order we are working, the d's and the λ's will be undergoing Brownian motion because the diagonal elements are moving in a random potential given by the values of the Q's. There are also additional contributions to the diffusion constant coming from the terms in Q that contribute to the eigenvalues at higher order. So we will have to be careful about the definitions of the velocities. In particular, we will have to recall that in the theory of stochastic processes the limits which define time derivatives are taken *after* the averages over probability distributions, not before. Therefore, as before, we must write

$$\int \mathrm{d}t \int \mathrm{d}d\, \mathrm{d}Q \rho_d(d)\rho_Q(Q) \frac{1}{2\mu} \left(\frac{\delta S}{\delta d_i^a}\right)^2$$

$$= \int \mathrm{d}t\, \mathrm{d}d\, \mathrm{d}Q \rho_d(d)\rho_Q(Q)\mu \big(V(d)_i^a\big)^2$$

$$= \int \mathrm{d}t \lim_{\Delta t\to 0} \int \mathrm{d}d\, \mathrm{d}Q \rho_d(d)\rho_Q(Q)\mu \left(\frac{\big[d_i^a(t+\Delta t) - d_i^a(t)\big]^2}{\Delta t^2}\right)$$

$$= \int \mathrm{d}t \lim_{\Delta t\to 0} \int \mathrm{d}d\, \mathrm{d}Q \rho_d(d)\rho_Q(Q) \frac{\mu}{2} \left\{ \left(\frac{\big[d_i^a(t+\Delta t) - d_i^a(t)\big]^2}{\Delta t^2}\right) + \left(\frac{\big[d_i^a(t) - d_i^a(t-\Delta t)\big]^2}{\Delta t^2}\right)\right\} .$$

Note that the last equation follows trivially, for smooth motion, but it will have non-trivial consequences once we have averaged over the Q's because the result for large N is to induce Brownian motion for the off-diagonal elements and eigenvalues. Now we perform the integral over the d's. It is useful to write

$$d_i^a = \lambda_i^a + \Delta\lambda_i^a \,, \tag{10.69}$$

where

$$\Delta\lambda_i^a(Q,\lambda) = -\sum_j \frac{Q_{ij}^a Q_{ji}^a}{\lambda_i^a - \lambda_j^a} + \cdots \tag{10.70}$$

has to be treated as a stochastic variable, taking into account its dependence on the Q's, which are themselves fluctuating due to the assumption that they are in equilibrium in a potential. We then have, to leading order in $1/N$,

$$I[\rho_d, S_d, T] = \int \mathrm{d}t \int \mathrm{d}\lambda \rho_d(\lambda,t) \int \mathrm{d}Q \rho_Q(Q) \left\{ \dot{S}(\lambda+\Delta\lambda, Q) + \frac{1}{2\mu}\left[\frac{\delta S(\lambda+\Delta\lambda,Q)}{\delta Q_{ij}^a}\right]^2 + U(\lambda,Q)\right\} + \text{K.E.} \,,$$

where, to leading order, the kinetic energy terms for the d's have become,

$$\text{K.E.} = \int \mathrm{d}t \lim_{\Delta t\to 0} \int \mathrm{d}\lambda \rho_d(\lambda,t) \int \mathrm{d}Q \rho_Q(Q) \frac{\mu}{2} \left\{ \frac{\big[\lambda_i^a(t+\Delta t) - \lambda_i^a(t)\big]^2}{\Delta t^2} + \frac{\big[\lambda_i^a(t) - \lambda_i^a(t-\Delta t)\big]^2}{\Delta t^2}\right\} .$$

We are now ready to integrate over the Q's. The key point is that the dependence of the λ's on the Q's through a sum of a large number of independent

terms, $\sum_j Q^a_{ij} Q^a_{ji}/(d^a_i - d^a_j)$, as well as the coupling of the λ's with the Q's coming from the terms in $U(\lambda, Q)$, turns the λ's into stochastic variables, described by a stochastic differential equation of the form

$$\mathrm{D}\lambda^a_i = b^a_i(\lambda, t)\mathrm{d}t + \Delta\lambda^a_i \ , \quad \Delta t > 0 \ , \tag{10.71}$$

$$\mathrm{D}\lambda^a_i = b^{*a}_i(\lambda, t)\mathrm{d}t + \Delta^*\lambda^a_i \ , \quad \Delta t < 0 \ , \tag{10.72}$$

with

$$\langle \Delta\lambda^a_i \Delta\lambda^b_j \rangle = \delta^{ab}\delta_{ij}\nu_\lambda \mathrm{d}t \ , \quad \mathrm{d}t > 0 \ , \tag{10.73}$$

$$\langle \Delta^*\lambda^a_i \Delta^*\lambda^b_j \rangle = -\delta^{ab}\delta_{ij}\nu_\lambda \mathrm{d}t \ , \quad \mathrm{d}t < 0 \ . \tag{10.74}$$

Here the brackets mean

$$\langle F(\lambda, Q) \rangle = \int \mathrm{d}Q \rho_Q(Q) F(\lambda, Q) \ . \tag{10.75}$$

We note that we can use the value of ν_λ given by (10.61), computed in the last section, as all the assumptions we made there have been carried over here, so long as we work to leading order in $1/N$ with T scaled as $T \approx 1/N$. We also have, from the Fokker–Planck equations, that the current velocity is

$$v^a_i(\lambda) = \frac{1}{2}(b^a_i + b^{*a}_i) \ , \tag{10.76}$$

while the osmotic velocity is

$$u^a_i(\lambda) = \frac{1}{2}(b^a_i - b^{*a}_i) = \nu_\lambda \frac{\delta \ln \rho_\lambda}{\delta \lambda^a_i} \ . \tag{10.77}$$

From these, we can derive

$$\lim_{\Delta t \to 0} \int \mathrm{d}Q \rho_d(\lambda, t)\rho_Q(Q) \frac{1}{2} \left(\frac{\left[\lambda^a_i(t + \Delta t) - \lambda^a_i(t)\right]^2}{\Delta t^2} \right) = \rho_d(\lambda, t)\left[b^a_i(\lambda, t)^2 + NC\right]$$

and

$$\lim_{\Delta t \to 0} \int \mathrm{d}Q \rho_d(\lambda, t)\rho_Q(Q) \frac{1}{2} \left(\frac{\left[\lambda^a_i(t) - \lambda^a_i(t - \Delta t)\right]^2}{\Delta t^2} \right) = \rho_d(\lambda, t)\left[b^{*a}_i(\lambda, t)^2 + NC\right] \ ,$$

where C is the infinite constant we defined in (10.21).

To go further we need to define the effective Hamilton–Jacobi function for the eigenvalues. We define

$$S_\lambda(\lambda) = \int \mathrm{d}Q \rho_Q(Q) S(\lambda, Q) \ . \tag{10.78}$$

We now show that, to leading order in $1/N$,

$$\mu v_i^a = \frac{\delta S_\lambda(\lambda)}{\delta \lambda_a^i} \ . \tag{10.79}$$

Consider the probability conservation law that follows from the statistical variational principle that defines the dynamics of our matrix model, viz., (10.62):

$$\dot{\rho}(d, Q) = -\frac{1}{\mu} \left\{ \frac{\delta}{\delta d_{ai}} \left[\rho(d, Q) \frac{\delta S(d, Q)}{\delta d_{ai}} \right] + \frac{\delta}{\delta Q_{aij}} \left[\rho(d, Q) \frac{\delta S(d, Q)}{\delta Q_{aij}} \right] \right\} \ . \tag{10.80}$$

But using (10.63) and (10.64), we have

$$\dot{\rho}(d, Q) = \dot{\rho}_d(d) \rho_Q(Q) \ . \tag{10.81}$$

By the same assumptions, since a thermal distribution is stationary and has no current velocity, we also have

$$v^{aij}(Q) = \frac{1}{\mu} \frac{\delta S(d, Q)}{\delta Q_{aij}} = O(1/N) \ . \tag{10.82}$$

Thus, integrating over the Q's,

$$\dot{\rho}_d(d) = -\frac{1}{\mu} \frac{\delta}{\delta d_{ai}} \left[\rho_d(d) \frac{\delta}{\delta d_{ai}} \int \mathrm{d}Q \rho_Q(Q) S(d, Q) \right] + O(1/N) \ . \tag{10.83}$$

To leading order we can replace the dependence on d_{ai} everywhere with dependence on λ_{ai}, since the terms by which they differ are also higher order in $1/N$. We then have

$$\dot{\rho}_d(\lambda) = -\frac{1}{\mu} \frac{\delta}{\delta \lambda_{ai}} \left[\rho_d(\lambda) \frac{\delta S_\lambda(\lambda)}{\delta \lambda_{ai}} \right] + O(1/N) \ . \tag{10.84}$$

But by (10.76), we must have

$$\dot{\rho}_d(\lambda) = -\frac{\delta \rho_d(\lambda) v^{ai}(\lambda)}{\delta \lambda_{ai}} \ . \tag{10.85}$$

This establishes (10.79). With this result we have the key relation that

$$\frac{\mu}{2}(b^2+b^{*2}) = \frac{\mu}{2}(v^2+u^2) = \left\{ \frac{1}{2\mu}\left[\frac{\delta S_\lambda(\lambda)}{\delta\lambda_i^a}\right]^2 + \frac{\mu\nu_\lambda^2}{2}\left[\frac{\delta \ln\rho_\lambda(\lambda)}{\delta\lambda_i^a}\right]^2 \right\} . \tag{10.86}$$

We also define

$$E_Q = \int \mathrm{d}Q\rho_Q(Q)\left\{ \frac{1}{2\mu}\left[\frac{\delta S_Q(Q)}{\delta Q_{ij}^a}\right]^2 + \frac{\mu\omega^2}{2}\mathrm{Tr}[Q,Q]^2 \right\} \tag{10.87}$$

and

$$\frac{\mu}{2}\Omega_d^2 \sum_{aij}(\lambda_i^a - \lambda_j^a)^2 = \int \mathrm{d}Q\rho_Q(Q)U^{\mathrm{int}}(\lambda,Q) . \tag{10.88}$$

We can estimate that $E_Q = TN(N-1)/4 \approx N\mu\omega^2$, so this is a divergent constant in the limit. The result is

$$I[\rho_d,S,T] = \int \mathrm{d}t \int \mathrm{d}\lambda \big[\rho_d(\lambda,t)\dot{S}_\lambda - H^{\mathrm{eff}}(S_\lambda,\rho_d,T)\big] , \tag{10.89}$$

where the effective Hamiltonian for the eigenvalues is

$$H^{\mathrm{eff}}(S_\lambda,\rho_\lambda,T) = \rho_d(\lambda)\Bigg(E_Q' + \left\{ \left[\frac{1}{2\mu}\frac{\delta S_\lambda(\lambda)}{\delta\lambda_i^a}\right]^2 + \frac{\mu\nu_\lambda^2}{2}\left[\frac{\delta\ln\rho_\lambda(\lambda)}{\delta\lambda_i^a}\right]^2 \right\} + \frac{\mu\Omega_d^2}{2}\sum_{aij}(\lambda_I^a - \lambda_j^a)^2 \Bigg)$$

and $E_Q' = E_Q + N\mu C$ contains both infinite constants. The resulting equations of motion are

$$E_Q' + \dot{S}_\lambda + \frac{1}{2\mu}\left[\frac{\delta S_d(\lambda)}{\delta\lambda_i^a}\right]^2 + \frac{\mu\Omega_d^2}{2}\sum_{aij}(\lambda_I^a - \lambda_j^a)^2 + U^{\mathrm{quantum}} = 0 , \tag{10.90}$$

and the current conservation equation

$$\dot{\rho}_\lambda = -\frac{1}{\mu}\partial^{ai}\rho_\lambda(\partial_{ai}S_\lambda) . \tag{10.91}$$

The so-called quantum potential is given by

$$\begin{aligned} U^{\mathrm{quantum}} &= \mu\nu_\lambda^2\left\{ \left[\frac{\delta\ln\rho_\lambda(\lambda)}{\delta\lambda_i^a}\right]^2 + \frac{1}{\rho_\lambda}\partial_{ai}(\rho_\lambda\partial^{ai}\ln\rho_\lambda) \right\} \\ &= -\mu\nu_\lambda^2\frac{1}{\sqrt{\rho_\lambda(\lambda)}}\nabla^2\sqrt{\rho_\lambda(\lambda)} . \end{aligned} \tag{10.92}$$

These we recognize as the real and imaginary parts of the Schrödinger equation, when we write

$$\Psi(\lambda,t)=\sqrt{\rho_\lambda}\mathrm{e}^{S_\lambda/\hbar}\ , \tag{10.93}$$

with

$$\hbar=\mu\nu_\lambda=\mu\omega\frac{t^{3/2}d}{4(d-1)^{3/2}}\ . \tag{10.94}$$

So, finally, we have in the limit $N\to\infty$,

$$\mathrm{i}\hbar\frac{\mathrm{d}\Psi(\lambda,t)}{\mathrm{d}t}=\left[-\frac{\hbar^2}{2\mu}\frac{\delta^2}{\delta(\lambda_i^a)^2}+\frac{\mu\Omega_d^2}{2}\sum_{aij}(\lambda_I^a-\lambda_j^a)^2+E'_Q\right]\Psi(\lambda,t)\ . \tag{10.95}$$

Finally, by repeating the argument in our Hamiltonian formulation on p. 135, we can show that the conserved energy of the original theory splits into two pieces:

$$H=H^\Psi+E'_Q\ , \tag{10.96}$$

where

$$H^\Psi=\int\mathrm{d}\lambda\bar\Psi\left[-\frac{\hbar^2}{2\mu}\frac{\delta^2}{\delta(\lambda_i^a)^2}+\frac{\mu\Omega_d^2}{2}\sum_{aij}(\lambda_I^a-\lambda_j^a)^2\right]\psi\ . \tag{10.97}$$

Since E'_Q is an infinite constant the result is that H^Ψ, which is the quantum mechanical energy, is conserved as $N\to\infty$. We can then renormalize the wavefunctional so that

$$\Psi_r(\lambda)=\mathrm{e}^{\mathrm{i}E'_Qt/\hbar}\Psi(\lambda)\ . \tag{10.98}$$

Finally, we note that as $\Omega_d^2\approx 1/N$ the eigenvalues become free in the limit $N\to\infty$. Thus, when $N\to\infty$, the probabilities evolve according to the free Schrödinger equation

$$\mathrm{i}\hbar\frac{\mathrm{d}\Psi_r(\lambda,t)}{\mathrm{d}t}=\left[-\frac{\hbar^2}{2\mu}\frac{\delta^2}{\delta(\lambda_i^a)^2}\right]\Psi_r(\lambda,t)\ . \tag{10.99}$$

10.7 Conclusion

We may summarize what has been done in the following way. To get near-diagonal behavior in our matrices, we take the whole system and raise it to a

finite temperature T. When we do that, we excite the off-diagonal elements. This potential is bounded from below and the minima are attained when the matrices are diagonal. So if we excite the off-diagonal elements, they can be thought of as N^2 objects moving around in this potential at low temperature. The trick is to take the value of N, the dimension of the matrices, to infinity, while taking the temperature to zero. What we have then is a kind of critical phenomenon exactly when we take the temperature to zero as $1/N$.

The key thing is that when N is large, each off-diagonal degree of freedom lives in a harmonic potential with some harmonic frequency which is related to the expectation value of the Q's, and that the expectation value of the Q's is in turn related to the temperature, which goes like $1/N$. When all this is brought together, the average size of the Q's goes like $1/\sqrt{N}$. The harmonic frequency turns out to go just like Ω^2, the fundamental coupling constant of the theory. The diagonal elements also live in harmonic potentials but it turns out that their frequency goes like $1/N$. They live in a harmonic potential created by all the elements, and basically there are N^2 off-diagonal elements which are moving around and creating all the noise. We may say that the diagonal elements are moving in a kind of a bath created by the off-diagonal elements. The harmonic frequency of the diagonal elements, however, turns out, when you scale it that way, to involve a factor of $1/N$. Therefore, in the limit $N \to \infty$, the harmonic frequency of the off-diagonal elements is constant, tuning the temperature like $1/N$, and the harmonic frequency of the diagonal elements goes to zero. We thus recover a situation in which there is no potential for the motion of the diagonal elements but there is a potential created by all the random motions for the off-diagonal elements. This in turn happens because there are more and more off-diagonal elements, even if we take the size of each one of them to zero as we take the temperature to zero.

Furthermore, because of the way that all the N^2 off-diagonal elements feed into the equations of motion for both the diagonal and off-diagonal elements, there is noise, so there are diffusion constants. What we see is that the diffusion constant for the diagonal elements also goes just like the coupling constant and does not go to zero or to infinity, but just stays constant as we take N to infinity. We are then left with Nelson's irreducible noise in that limit.

The idea is that the off-diagonal elements have effectively decoupled. They are moving amongst themselves in some complicated, ever-changing harmonic potential. The diagonal elements do not feel a potential, but only noise. Indeed, the strange thing about Nelson's hypothesis was that we have a source of noise which does not violate the conservation of energy averaged over the ensemble. So the idea was to generate a system where we could explain a source of noise which moved the classical variables around without systematically draining or inputting energy. In fact, we have reformulated Nelson's idea, without doing anything new apart from the technical items detailed above.

What we have shown here may be summarized by saying that matrix theory may not only give rise to string theory, and hence gravity, but it may also give rise to quantum theory, in the sense that the quantum evolution of the eigenvalues may appear, at large N, to be a consequence of the classical statistical physics of the matrices. We have shown that we can obtain a formula for $\hbar$ in terms of the fundamental parameters of a more fundamental, hidden variables theory [see (10.94)].

This means that we may be able to solve the daunting conceptual problems of quantum theory by means of a simple physical hypothesis: that the theory of gravity and hence spacetime arises from a non-local background-independent theory in which geometry initially plays no role and the physical degrees of freedom represent relational rather than intrinsic properties. There remain many open questions. A short list is:

- Is it possible to extend these results to the supersymmetric models that arise in string theory [7]?
- Is it possible that the existence of dualities that connect certain quantum field theory observables to the classical limit of string theory are related to the fact that the classical matrix theory can in a certain limit reproduce a quantum theory?
- How does Lorentz invariance and relativistic causality arise in the matrix models which describe relativistic string and membrane theories, and how is this compatible with the non-local dynamics of the off-diagonal elements?
- Can this be extended to truly background-independent matrix models, such as the cubic matrix models and the matrix models which have been developed for spin foams [3]?
- Are there any practical experimental predictions that follow from these theories?

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11 Towards a General Operational and Realistic Framework for Quantum Mechanics and Relativity Theory

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We propose a general operational and realistic framework that aims at a generalization of quantum mechanics and relativity theory, such that both appear as special cases of this new theory. Our framework is operational, in the sense that all aspects are introduced with specific reference to events to be experienced; and realistic, in the sense that the hypothesis of an independent existing reality is taken seriously.

To reach this framework we present a detailed study of standard quantum mechanics within the axiomatic approach to quantum mechanics, more specifically the Geneva–Brussels approach, identifying two of the traditional 6 axioms as 'failing axioms'. We prove that these two failing axioms are at the origin of the impossibility for standard quantum mechanics to describe a continuous change from quantum to classical and hence its inability to describe macroscopic physical reality. Moreover we show that the same two axioms are also at the origin of the impossibility for standard quantum mechanics to deliver a model for the compound entity of two 'separated' quantum entities. We contend that these two axioms must be replaced in order to proceed to the more general theory.

Next we analyze the nature of the quantum probability model and show that it can be interpreted as a consequence of a lack of knowledge about the interaction between the measurement apparatus and the physical entity under consideration. These two insights, the failing axioms and the nature of quantum probability, give rise to a very specific view concerning the quantum phenomenon of nonlocality. Nonlocality should be interpreted as nonspatiality. This means that an entity in a nonlocal state, such as the typical EPR state, is not 'inside space'. As a consequence, space is no longer the all-embracing theatre of reality, but a structure that has emerged together with the macroscopic material entities that have emerged from the microworld. This clarifies why general relativity theory cannot be used as a basis for the elaboration of the new generalized theory, since in general relativity theory the set of events is taken a priori to be the time–space continuum. Hence, in general relativity theory, time–space is a basic structure considered to capture all of reality. In our framework we introduce 'happenings' and the 'set of happenings' as constituting reality. However, a happening is not identified with a point of time–space, as is the case for an event in general relativity

theory. We analyze different aspects of the new framework, and list the most important problems to be investigated if this framework is to be made into a workable theory which is as complete as possible.

11.1 Introduction

Quantum mechanics, even after so many years of reflection by the brightest scientists of our day, still confronts us with really fundamental problems. While microscopic effects predicted by it have been experimentally verified, they remain irreconcilable with macroscopic reality. Moreover some of these effects are incompatible with the other major theory of modern physics, namely, relativity, thereby introducing a deep schism at the very basis of mainstream science. In this article we focus on the following question: What kind of theory can be envisaged to replace quantum mechanics as a better description of physical reality?

We believe that very fundamental changes will take place from a theoretical point of view in the decades to come, and that what is now referred to as standard quantum mechanics[1] will be looked upon as a special case of a more general theory still to be developed. When this new theory exists, we will then also understand why standard quantum mechanics gave such good predictions for experiments performed with quantum entities. Of course, general relativity should also appear in some way as a special case of the new theory to be developed.

One of the reasons why it is so difficult to develop the new theory is that general relativity and quantum mechanics are fundamentally different theories as regards the way they were constructed. General relativity is a masterpiece of conscious construction starting from deep physical principles and simple but very general ideas, such as the equivalence of gravitational and inertial mass, and, from a foundational standpoint, was mainly the work of one person, Albert Einstein. Quantum mechanics, however, has grown out of a complex configuration of problems, guided by strange metaphors, such as the wave–particle duality, and abstract mathematics, such as matrix calculus and the Hilbert space formalism. Hence, it is no wonder that the theories are basically very different and very difficult to integrate.

The approach to quantum mechanics that we have been elaborating, and that will also be the inspiration for our ideas on the future of quantum mechanics and the nature of the new theory, is not the most commonly known approach. There are several reasons for this, but one of them is certainly that it is very different from the approaches that start more straightforwardly from quantum mechanics and/or relativity theory as they are formulated in their standard form. In the sections to follow we will outline our approach and also

[1] We call the theory formulated by John von Neumann in 1934 standard quantum mechanics [1].

point out why we think that it contains a great potential to help in generating the framework for a new theory that could integrate relativity theory and quantum mechanics.

11.2 Operational Axiomatic Quantum Mechanics

The framework that we develop has roots in the work of John von Neumann, in collaboration with Garett Birkhoff, that is almost as old as the standard formulation of quantum mechanics itself [2]. Indeed, even during the first years of quantum mechanics, the formalism that is now referred to as standard quantum mechanics [1] was thought to be too specific by the founding fathers themselves. One of the questions that was obviously at the origin of this early dissatisfaction is: Why would a complex Hilbert space deliver 'the' unique mathematical structure for a complete description of the microworld? Would that not be amazing? What is so special about a complex Hilbert space that its mathematical structure would play such a fundamental role?

Let us turn for a moment to general relativity to raise a suspicion about the fundamental role of the complex Hilbert space for quantum mechanics. General relativity is founded on the mathematical structure of Riemann geometry. In this case, however, it is much more plausible that the right mathematical structure has indeed been adopted. Riemann developed his theory as a synthesis of the work of Gauss, Lobaskjevski and Bolay on non-Euclidean geometry, and his aim was to work out a theory for the description of the geometrical structure of the world in all its generality. Hence, Einstein had recourse to the work of Riemann to express his ideas and intuitions on spacetime and its geometry and this lead to general relativity. General relativity could be called in this respect the geometrization of a part of the world including gravitation.

There is, of course, a definite reason why von Neumann used the mathematical structure of a complex Hilbert space for the formalization of quantum mechanics, but this reason is much less profound than it is for Riemann geometry and general relativity. The reason is that Heisenberg's matrix mechanics and Schrödinger's wave mechanics turned out to be equivalent, the first being a formalization of the new mechanics making use of l_2, the set of all square summable complex sequences, and the second making use of $L_2(\mathbb{R}^3)$, the set of all square integrable complex functions of three real variables. The two spaces l_2 and $L_2(\mathbb{R}^3)$ are canonical examples of a complex Hilbert space. This means that Heisenberg and Schrödinger were already working in a complex Hilbert space, when they formulated matrix mechanics and wave mechanics, without being aware of it. This made it a straightforward choice for von Neumann to propose a formulation of quantum mechanics in an abstract complex Hilbert space, reducing matrix mechanics and wave mechanics to two specific cases.

One problem with the Hilbert space representation was known from the start. A (pure) state of a quantum entity is represented by a unit vector or

ray of the complex Hilbert space, and not by a vector. Indeed vectors contained in the same ray represent the same state or one has to renormalize the vector that represents the state after it has been changed in one way or another. It is well known that if rays of a vector space are called points and two-dimensional subspaces of this vector space are called lines, the set of points and lines corresponding in this way to a vector space form a projective geometry. What we just noted about the unit vector or ray representing the state of the quantum entity means that in some way the projective geometry corresponding to the complex Hilbert space represents the physics of the quantum world in a more intrinsic way than the Hilbert space itself. This state of affairs is revealed explicitly in the dynamics of quantum entities, built up using group representations, and one has to consider projective representations, which are representations in the corresponding projective geometry, and not vector representations [3].

The title of the article by John von Neumann and Garett Birkhoff [2] that we mentioned as the founding article for our approach is *The Logic of Quantum Mechanics*. Let us explain briefly what Birkhoff and von Neumann do in this article. First of all, they note that an operational proposition of a quantum entity is represented in the standard quantum formalism by an orthogonal projection operator or by the corresponding closed subspace of the Hilbert space $\mathcal{H}$. Let us denote the set of all closed subspaces of $\mathcal{H}$ by $\mathcal{P}(\mathcal{H})$. Next Birkhoff and von Neumann show that the structure of $\mathcal{P}(\mathcal{H})$ is not that of a Boolean algebra, the archetypical structure of the set of propositions in classical logic. More specifically, it is the distributive law between conjunction and disjunction

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c) \,, \tag{11.1}$$

which is not necessarily valid for the case of quantum propositions $a, b, c \in \mathcal{P}(\mathcal{H})$. A whole line of research, called quantum logic, was born as a consequence of the Birkhoff and von Neumann article. The underlying philosophical idea is that, in the same manner as general relativity has introduced non-Euclidean geometry into the reality of the physical world, quantum mechanics introduces non-Boolean logic. The quantum paradoxes would be due to the fact that we reason with Boolean logic about situations with quantum entities, while these situations should be reasoned about with non-Boolean logic.

Although fascinating as an idea and worth taking seriously [4], it is not this idea that is at the origin of our approach. A more important aspect of what Birkhoff and von Neumann did in their article, not to be found in the title, is that they shifted attention to the mathematical structure of the set of operational propositions $\mathcal{P}(\mathcal{H})$ instead of the Hilbert space $\mathcal{H}$ itself. In this sense it is important to pay attention to the fact that $\mathcal{P}(\mathcal{H})$ is the set of all operational propositions, i.e., the set of yes/no experiments on a quantum entity. They opened a way to connect abstract mathematical concepts

of the quantum formalism, namely the orthogonal projection operators or closed subspaces of the Hilbert space, directly to physical operations in the laboratory, namely the yes/no experiments.

George Mackey followed this idea when he wrote his book on the mathematical foundations of quantum mechanics [5]. He started from the other end and considered as a basis the set $\mathcal{L}$ of all operational propositions, meaning propositions that are testable by yes/no experiments on a physical entity. He then introduced as the axiom that this set $\mathcal{L}$ has to have a structure isomorphic to the set of all closed subspaces $\mathcal{P}(\mathcal{H})$ of a complex Hilbert space in the case of a quantum entity. He stated that it would be interesting to invent a set of axioms on $\mathcal{L}$ that would gradually make $\mathcal{L}$ more and more like $\mathcal{P}(\mathcal{H})$, so as to arrive eventually at an isomorphism when all the axioms were satisfied. While Mackey was writing his book, such work was already underway. A year later Constantin Piron proved a fundamental representation theorem. Starting from the set $\mathcal{L}$ of all operational propositions of a physical entity and introducing five axioms on $\mathcal{L}$, he proved that $\mathcal{L}$ is isomorphic to the set of closed subspaces $\mathcal{P}(V)$ of a generalized Hilbert space V whenever these five axioms are satisfied [6]. Although we do not want to get too technical, we have to elaborate on some of the aspects of this representation theorem to be able to explain our framework.

We have already mentioned that Birkhoff and von Neumann had noticed that the set of closed subspaces $\mathcal{P}(\mathcal{H})$ of a complex Hilbert space $\mathcal{H}$ is not a Boolean algebra, because distributivity between conjunction and disjunction is not satisfied [see (11.1)]. However, the set of closed subspaces of a complex Hilbert space does form a lattice, which is a more general mathematical structure than a Boolean algebra. Moreover, a lattice where the distributivity rule (11.1) is satisfied is a Boolean algebra, which indicates that the lattice structure is the one to consider for the quantum mechanical situation. As we will see in more detail later, and to make the connection with general relativity once again, the lattice structure is indeed to a Boolean algebra what general Riemann geometry is to Euclidean geometry. And in the process, we have understood why the structure of operational propositions of the world is not a Boolean algebra but a lattice. This is strictly due to the fact that measurements can have an uncontrollable influence on the state of the physical entity under consideration. We will explain this insight in detail later on, but mention it already now, so that it is clear that the intuition of Birkhoff and von Neumann, and later Mackey, Piron and others, although only mathematical intuition at that time, was correct.

When Piron proved his representation theorem in 1964, he concentrated on the lattice structure for the formulation of the five axioms. Meanwhile much more research has been done, motivated both physically, in an attempt to make the approach more operational, and mathematically, in trying to get axiomatically closer to the complex Hilbert space. In the presentation of our framework, we give the most recent update of it, and hence somewhat

neglect the original formulation, e.g., when we explain the representation theorem due to Piron.

But before outlining our approach, we want to explain why we think that this approach holds the potential to generate the framework for the new theory to be developed, generalizing quantum mechanics and relativity theory. General relativity is a theory that brings part of the world, which in earlier Newtonian mechanics was classified within dynamics, into the geometrical realm of reality. More specifically, it confronts us with the pre-scientific and naive realistic vision of space, time, matter and gravitation. It teaches us in a deep and new way, compared to Newtonian physics, what things exist, how they exist, how they are related, and how they influence each other.

But there is one striking deficiency in relativity theory: it does not take into account the influence of the observer, the effect that the measuring apparatus has on the thing observed. It does not confront the subject–object problem and its influence on reality. It cannot do this because its mathematical apparatus is based on the Riemann geometry of time–space, hence introducing the prejudice that time–space is actually there, filled up with fields and matter, which are also there independently of the observer. There is no fundamental role for the creation of 'new' within relativity theory, everything just 'is' and we are only there to 'detect' how this everything 'is'.

This is also the reason why general relativity can easily be interpreted as delivering a model for the whole universe, whatever this may mean. We know that quantum mechanics takes into account in an essential way the effect of the observer through the measuring apparatus on the state of the physical entity under study. In the new theory to be developed, this effect should certainly also appear in a fundamental way. We believe that general relativity has explored in great depth the question of how things can 'be' in the world. The theory that we develop explores in great depth the question of how things can 'act' in the world. And it does explore this question of 'action in the world' in a very similar manner to the way general relativity theory does with its question of 'being of the world'. This means that our approach can be seen as the development of a general theory of actions in the world, in the same manner that Riemann's approach can be seen as a general theory of 'geometrical forms existing in the world'. Of course Riemann geometry is not equivalent to general relativity. A lot of detailed physics had to be known to apply Riemann geometry before it could lead to general relativity. This is the same with our approach: it has the potential to deliver the framework for the new theory, in a similar way to the way that Riemann geometry had the potential to deliver the framework for general relativity.

We note that in principle a theory that describes the possible actions in the world, and a theory that delivers a model for the whole universe, should not be incompatible. It should even be that the theory that delivers a model of the whole universe should incorporate the theory of actions in the world, which would mean for the situation that exists now, general relativity should contain

quantum mechanics, if it really delivers a model for the whole universe. That is why we believe that Einstein's attitude, trying to incorporate the other forces and interactions within general relativity, contrary to common belief, was the right one, globally speaking. What Einstein did not know at that time was the reality of nonlocality in the micro-world. From our approach it follows that nonlocality should be interpreted as nonspatiality, which means that the reality of the micro-world, and hence the reality of the universe as a whole, is not time–space-like. Time–space is not the global theatre of reality, but rather a crystallization and structuration of the macro-world. Time–space has come into existence together with the macroscopic material entities, and hence it is 'their' time and space, but it is not the theatre of the microscopic quantum entities.

This fact is the fundamental reason why general relativity, built on the mathematical geometrical Riemannian structure of time–space, cannot be the canvas on which the new theory can be painted. A way to express this technically would be to say that the set of events cannot be identified with the set of time–space points, as is done in relativity theory. One must have recourse to a theory that describes reality as a kind of pre-geometry, and where the geometrical structure arises as a consequence of interactions that collapse into the time–space context. We think that the approach that we develop can deliver the framework as well as the methodology to construct and elaborate such a theory. This is in our opinion the most fundamental role that quantum mechanics, or better the generalizations of quantum mechanics in the spirit of our approach, because we believe that standard quantum mechanics is mathematically too specific and too constrained to play this role, will play in the decades to come. In the next section we introduce the basic objects making up our approach.

11.3 State Property Spaces

Mackey and Piron introduced the set of yes/no experiments but then immediately shifted to an attempt to axiomatize the lattice of (operational) propositions of a quantum entity mathematically. Mackey immediately postulated an isomorphism with $\mathcal{P}(\mathcal{H})$ and Piron gave five axioms to come as close as possible to $\mathcal{P}(\mathcal{H})$. Furthermore, Piron's axioms are mostly motivated by mimicking the structure of $\mathcal{P}(\mathcal{H})$ mathematically. In later work Piron made a more determined attempt to found some of the axioms operationally [7], and this attempt was worked out further in [8–10], to arrive at a full operational foundation only recently [11–14].

Mathematically the circle was closed only recently. At the time when Piron gave his five axioms that lead to the representation within a generalized Hilbert space, there only existed three examples of generalized Hilbert spaces that fitted all the axioms, namely real, complex and quaternionic Hilbert

space, also referred to as the three standard Hilbert spaces.[2] Years later, Hans Keller constructed the first counterexample, more specifically an example of an infinite-dimensional generalized Hilbert space that is not isomorphic to one of the three standard Hilbert spaces [15].

The study of generalized Hilbert spaces, nowadays also called orthomodular spaces, developed into a research subject of its own, and recently Maria Pia Solèr proved a groundbreaking theorem in this field. She proved that an infinite-dimensional generalized Hilbert space that contains an orthonormal basis is isomorphic with one of the three standard Hilbert spaces [16]. Meanwhile it has also been possible to formulate an operational axiom, called 'plane transitivity' on the set of operational propositions that implies Solèr's condition [17], which completes the axiomatics for standard quantum mechanics by means of six axioms, the original five axioms of Piron and plane transitivity as the sixth axiom.

Let us now explain the operational axiomatic approach to quantum mechanics that we develop in its most recent version. Operational propositions have meanwhile been called properties. Hence the basic things to consider for a physical entity S are:

- its set of states Σ, where we denote states by symbols $p, q, r, \ldots$,
- its set of properties $\mathcal{L}$, where we denote properties by symbols $a, b, c, \ldots$,
- a relation of 'actuality' between the states and properties that expresses the basic statement: the property $a \in \mathcal{L}$ is actual if the entity is in state $p \in \Sigma$. This we do by introducing a function $\kappa : \mathcal{L} \to \mathcal{P}(\Sigma)$ such that $\kappa(a)$ is the set of all states of the entity that make property a actual, and κ is called the Cartan map.[3] The basic relation between states and properties, viz., the property $a \in \mathcal{L}$ is actual if the entity is in state $p \in \Sigma$, is then equivalent to the mathematical expression $p \in \kappa(a)$.

The triple $(\Sigma, \mathcal{L}, \kappa)$ is called a state property space [12].

Definition 1 (State Property Space). *The triple $(\Sigma, \mathcal{L}, \kappa)$, called a state property space, consists of two sets Σ and $\mathcal{L}$, where Σ is the set of states of a physical entity S, and $\mathcal{L}$ its set of properties, and a function $\kappa : \mathcal{L} \to \mathcal{P}(\Sigma)$, called the Cartan map, such that for $a \in \mathcal{L}$, we have that $\kappa(a)$ is the set of states that make a actual.*

[2] There do exist a lot of finite-dimensional generalized Hilbert spaces that are different from the three standard examples. But since a physical entity has to have at least a position observable, it follows that the generalized Hilbert space must be infinite-dimensional. At the time of Piron's representation theorem, the only infinite-dimensional cases that were known were the three standard Hilbert spaces, over the real, complex or quaternionic numbers.

[3] The idea of characterizing properties by the set of states that make them actual can be found in the work of E. Cartan, and that is why we have called this function the Cartan map [8].

The state property space will be the basic mathematical structure that we start with. It can easily be completely operationally founded in the following way. For each property $a \in \mathcal{L}$, we suppose that there is a yes/no experiment α that tests this property. This means the following: a state $p \in \Sigma$ is contained in $\kappa(a)$ iff the outcome for the yes/no experiment α is yes with certainty. If $p \notin \kappa(a)$, then the outcome for α is uncertain (can be yes or no).

There exist two natural pre-order relations,[4] one on $\mathcal{L}$ and one on Σ, defined as follows:

Definition 2 (Pre-Order Relations). *Suppose that $(\Sigma, \mathcal{L}, \kappa)$ is the state property space describing the physical entity S. For $a, b \in \mathcal{L}$ and $p, q \in \Sigma$, we define:*

$$a < b \Longleftrightarrow \kappa(a) \subset \kappa(b) \ , \tag{11.2}$$

$$p < q \Longleftrightarrow \forall a \in \mathcal{L} : q \in \kappa(a) \Longrightarrow p \in \kappa(a) \ , \tag{11.3}$$

and say 'a implies b' if $a < b$, and 'p implies q' if $p < q$.

The physical meaning of these two pre-order relations is obvious, for example $a < b$ means that, whenever a is actual, then b is also actual.

Let us see how the mathematical structure of a state property space is present in classical mechanics as well as in quantum mechanics. For a classical entity described by classical mechanics, the set of states is the state space Ω, and for a quantum entity described by standard quantum mechanics, the set of states is the set of unit vectors of the complex Hilbert space $\mathcal{H}$, which we denote by $\Sigma(\mathcal{H})$. We have already mentioned that a property of a quantum entity described by standard quantum mechanics is represented by the closed subspace which is the range of the projection operator that describes the yes/no experiment testing this property. Hence $\mathcal{L}$ equals $\mathcal{P}(\mathcal{H})$, the set of all closed subspaces of $\mathcal{H}$. For a classical entity described by classical mechanics, each subset of the state space represents a property, which shows that $\mathcal{L}$ equals $\mathcal{P}(\Omega)$, the set of all subsets of Ω. The Cartan map κ in the classical case is the identity, and in the quantum case it is the function that maps a closed subspace onto the set of unit vectors contained in this closed subspace. To conclude, classical mechanics has a state property space $\big(\Omega, \mathcal{P}(\Omega), \kappa\big)$, where $\kappa(A) = A$, and quantum mechanics has a state property space $\big(\Sigma(\mathcal{H}), \mathcal{P}(\mathcal{H}), \kappa\big)$, where $\kappa(A) = \{u | u \in \Sigma(\mathcal{H}), u \in A\}$.

By means of these two examples, we can already see that the shift of attention introduced by Birkhoff and von Neumann, Mackey, Piron and others to the set of operational propositions (called properties now), and hence mathematically the shift of attention from $\mathcal{H}$ to $\mathcal{P}(\mathcal{H})$ for a quantum entity, and the shift of attention from Ω to $\mathcal{P}(\Omega)$ for a classical entity, makes it possible to

[4] A pre-order relation $<$ is a relation that is reflexive ($x < x$), and transitive ($x < y$ and $y < z$ implies $x < z$).

consider a classical entity and a quantum entity within the same formalism. The state space of a classical physical entity is a very different mathematical structure from the complex Hilbert space of a quantum mechanical physical entity, but the sets of properties of both entities are just variations on a similar mathematical structure. This means that this approach has the potential to eventually understand the difference between classical and quantum, and we thus have a real possibility of working out a theory that integrates both descriptions, and where both classical mechanics and quantum mechanics appear as special cases.

11.4 The Axioms

The state property space $(\Sigma, \mathcal{L}, \kappa)$ is a purely operational structure, which means that no axioms are necessary to get it. Any physical entity has it. To come closer to the two cases, classical mechanics and quantum mechanics, we have to introduce axioms. Some of them will have a well-defined operational interpretation and others will be of a purely technical mathematical nature.

11.4.1 Axiom 1: State Property Determination

The first axiom amounts to demanding that:

- the set of states that make a certain property actual determine this property,
- the set of properties that are actual in a certain state determine this state.

That is why we call it the axiom of state and property determination. It is satisfied both in standard quantum mechanics and in classical mechanics. Let us formulate it:

Axiom 1 (State Property Determination). *Suppose we have a physical entity S described by a state property space $(\Sigma, \mathcal{L}, \kappa)$. The first axiom of state and property determination is satisfied iff for $p, q \in \Sigma$ and $a, b \in \mathcal{L}$, we have*

$$\kappa(a) = \kappa(b) \Rightarrow a = b \,, \tag{11.4}$$

$$\{a | a \in \mathcal{L}, p \in \kappa(a)\} = \{a | a \in \mathcal{L}, q \in \kappa(a)\} \Longrightarrow p = q \,. \tag{11.5}$$

It is easy to check that if Axiom 1 is satisfied, the two pre-order relations of Definition 2 become partial order relations.[5] From a mathematical point of view, an obvious question concerning a partially ordered set is whether there exist infima and suprema for this partial order relation. From a physical

[5] A partial order relation $<$ is a pre-order relation which, apart from being reflexive and transitive, is also symmetric ($x < y$ and $y < x$ implies $x = y$).

operational point of view, for two properties $a, b \in \mathcal{L}$, the infimum, which we denote $a \wedge b$ if it exists, would normally play the role of the conjunction of the two properties a and b, hence the property a 'and' b. The supremum, which we denote $a \vee b$ if it exists, would normally play the role of the disjunction of the two properties a and b, hence the property a 'or' b. For states, the meaning of the infimum and supremum is less straightforward, but we will see later that we do not have to bother about this, because another axiom is satisfied, more specifically Axiom 2, that makes the question irrelevant. So let us concentrate on the structure of $\mathcal{L}$, the set of properties of the physical entity S, which is now a partially ordered set, since we suppose Axiom 1 to be satisfied, with partial order relation $<$ as introduced in Definition 2.

We arrive here at the first aspect of quantum mechanics, related to the existence of the superposition principle, that can be understood and explained by means of our approach.

11.4.2 Conjunctions and Disjunctions

It turns out that, if we take into account that the general physical operational situation for two measurements is the situation where they cannot necessarily be carried out at once (or together), which in quantum jargon means that they are incompatible, we can show that in this case, the conjunction for these properties, of which one is measured by one of the measurements and the other by the other measurement, still exists as an operational property, but the disjunction does not necessarily exist as an operational property. This is a subtle matter and one that is not easy to explain in a few words, but we will make an attempt.

Suppose that we have two yes/no experiments α and β testing properties a and b. If we think of the standard way to define conjunction and disjunction in logic, by means of truth tables, it is obvious that both conjunction and disjunction can only be defined operationally if both yes/no experiments can be performed together, because that is the only way to form the truth tables operationally. For the conjunction, the outcome (yes, yes) for the joint measurement of α and β is replaced by 'yes', and the outcomes (yes, no), (no, yes) and (no, no) are replaced by 'no', while for the disjunction the outcomes (yes, yes), (yes, no) and (no, yes) are replaced by 'yes', while the outcome (no, no) is replaced by 'no'. However, this procedure cannot be applied when the two yes/no measurements α and β cannot be performed together.

The subtlety of the matter is that for the conjunction there is another procedure available which can always be applied, while for the disjunction this is not the case. To make this clear, consider for the two yes/no experiments α and β the yes/no experiment $\alpha \cdot \beta$, which we call the product experiment, which consists in choosing (at random or otherwise) one of the yes/no experiments, α or β, performing the chosen experiment and giving the outcome, yes or no, that occurs in this way, to the product yes/no experiment $\alpha \cdot \beta$. When

will $\alpha \cdot \beta$ give the outcome yes with certainty? Obviously, if and only if both α and β give the outcome yes with certainty. This means that $\alpha \cdot \beta$ tests the property a 'and' b. Since $\alpha \cdot \beta$ always exists, even if α cannot be performed together with β, because we only have to choose α or β to perform it, this proves that for two operational properties a and b, the property a 'and' b always exists as an operational property.

In [8], we gave an example where this is particularly evident. Consider a piece of wood and two properties a and b of the piece of wood, where a is the property 'the piece of wood burns well' and b is the property 'the piece of wood floats on water'. The yes/no experiment α, testing a, consists in putting the piece of wood on fire following a well-defined procedure, and seeing whether it burns. If so, the outcome 'yes' occurs. The yes/no experiment β, testing b, consists in putting the piece of wood on water and seeing whether it floats. If so, the outcome 'yes' occurs. Obviously it is rather difficult to perform both yes/no experiments together, and if one did try to do so, no reliable outcome would occur. But we all agree that there exist a lot of pieces of wood for which both properties a and b are actual at once. The reason that we are all convinced of this fact is that we unconsciously use the yes/no experiment $\alpha \cdot \beta$ to test the conjunction property a 'and' b. Indeed, we decide that, for a specific piece of wood in a specific state, both properties are actual, because if we chose to perform one of the two yes/no experiments α or β, the outcome 'yes' would occur with certainty. This is exactly the same as performing $\alpha \cdot \beta$, the product yes/no experiment.

Hence, the reason why there is an asymmetry for the existence of an operational conjunction and disjunction, the conjunction always existing, while the disjunction only exists when the corresponding experiments can be performed together, is because, to perform the product experiment $\alpha \cdot \beta$ of two experiments α and β, we only need to be able to perform α or β, which is indeed always possible. We note that the product yes/no experiment exists for any number of yes/no experiments $\{\alpha_i\}_i$, and operationally defines the conjunction of all the corresponding properties $\{a_i\}_i$. It can be proven that the conjunction is an infimum for the partial order relation $<$ existing on $\mathcal{L}$, and that is why we denote it by $\wedge_i a_i$. It is a mathematical theorem that a partially ordered set $\mathcal{L}, <$ such that for any family of elements $\{a_i\}_i$ there exists an infimum, is a complete lattice[6] if there exists a maximal element of $\mathcal{L}$. And this maximal element exists, e.g., the property 'the physical entity under consideration exists', which we denote by I, is such a maximal element of $\mathcal{L}, <$. The supremum for a family of elements $\{a_i\}_i$, denoted $\vee_i a_i$, is then defined mathematically by the formula

$$\bigvee_i a_i = \bigwedge_{x \in \mathcal{L}, a_i < x \ \forall i} x \ . \tag{11.6}$$

[6] A complete lattice is a partially ordered set such that, for any family of elements, there exists an infimum and a supremum for this partial order.

As a conclusion we can say that, if Axiom 1 is satisfied, the set of properties $\mathcal{L}$ of the state property space $(\Sigma, \mathcal{L}, \kappa)$ describing the physical entity S is a complete lattice. Let us note that, as a consequence, $\mathcal{L}$ also contains a minimal element, which we denote by 0.

We promised that we would be able to explain something related to the superposition principle in quantum mechanics by what we have been analyzing in the foregoing. To do this, let us return to the state property spaces $\big(\Omega, \mathcal{P}(\Omega), \kappa\big)$ of classical mechanics and $\big(\Sigma(\mathcal{H}), \mathcal{P}(\mathcal{H}), \kappa\big)$ of quantum mechanics. In $\mathcal{P}(\Omega)$ the infimum and supremum of two subsets $A, B \in \mathcal{P}(\Omega)$ are given respectively by the intersection $A \cap B \in \mathcal{P}(\Omega)$ and the union $A \cup B \in \mathcal{P}(\Omega)$ of subsets. For the case of $\mathcal{P}(\mathcal{H})$, the infimum of two closed subspaces $A, B \in \mathcal{P}(\mathcal{H})$ is given by the intersection $A \cap B \in \mathcal{P}(\mathcal{H})$, because the intersection of two closed subspaces is indeed another closed subspace. On the other hand, the union of two closed subspaces is not in general a closed subspace. This means that the union does not give us the supremum in this case. For two closed subspaces $A, B \in \mathcal{P}(\mathcal{H})$, the smallest closed subspace that contains both is $\overline{A + B}$, the topological closure of the sum of the two subspaces. Hence this is the supremum of A and B in $\mathcal{P}(\mathcal{H})$. The vectors contained in the topological closure $\overline{A + B}$ of the sum of A and B are exactly the vectors that are superpositions of vectors in A and vectors in B. Hence, for a quantum entity, described by $\big(\Sigma(\mathcal{H}), \mathcal{P}(\mathcal{H}), \kappa\big)$, there are additional vectors in the supremum of A and B, contained neither in A nor in B, while for the classical entity, described by $\big(\Omega, \mathcal{P}(\Omega), \kappa\big)$, there are no such additional elements, because the supremum of A and B is the union $A \cup B$. This is due to the fact that quantum mechanics experiments that cannot be performed together are an essential ingredient of the theory, while for classical mechanics, although such experiments exist, think of the example of the piece of wood, they can always be substituted by other experiments that can be performed together. For the piece of wood we can, for example, break the sample in two pieces, putting one in water and the other in fire.

The role of the superposition principle is still a little more subtle than we have explained here, because even for compatible properties, for example, two closed subspaces A and B with respective projection operators that commute, there exist states that are contained neither in A, nor in B, superposition states of states in A and B, that are contained in the closure of the sum $\overline{A + B}$. This is due to the possibility of EPR-like correlations in quantum mechanics. We have analyzed this effect in detail in [18], but can explain the crux of it in a few lines. If, for example, in performing the yes/no experiments α and β, testing properties a and b, there is an EPR-type correlation, such that (yes, no) or (no, yes) always comes out for the experiment that performs both α and β together, then a 'or' b is actual as a property, following the rules of the truth tables, but neither a nor b is actual, because both (yes, no) or (no, yes) are possible outcomes. This shows that, in the presence of

EPR-like correlations, the 'or' defined by the truth table is not the 'or' of classical logic.

11.4.3 Axiom 2: Atomisticity

An element of a (complete) lattice is called an atom if it is a smallest element different from the minimal element 0. Let us define precisely what we mean.

Definition 3 (Atom of a Complete Lattice). *We say that $s \in \mathcal{L}$ is an atom of $\mathcal{L}$ if, for $x \in \mathcal{L}$, we have:*

$$0 < x < s \Longrightarrow x = 0 \text{ or } x = s \, . \tag{11.7}$$

The atoms of $\big(\mathcal{P}(\Omega), \subset, \cap, \cup\big)$ are the singletons of the phase space Ω, and the atoms of $\big(\mathcal{P}(\mathcal{H}), \subset, \cap\big)$ are the one-dimensional subspaces (rays) of the Hilbert space $\mathcal{H}$.

The second axiom consists in demanding that the states can be considered as atoms of the property lattice.

Axiom 2 (Atomisticity). *Suppose that we have a physical entity S described by a state property space $(\Sigma, \mathcal{L}, \kappa)$. For $p \in \Sigma$, we have that*

$$\bigwedge_{p \in \kappa(a)} a \tag{11.8}$$

is an atom of $\mathcal{L}$.

This axiom is obviously satisfied in classical mechanics as well as in quantum mechanics.

11.4.4 Axiom 3: Orthocomplementation

The third axiom introduces an orthocomplementation structure into the lattice of properties. At first sight this orthocomplementation could be seen as a structure that plays a similar role for properties to the one that negation plays for propositions in logic. But that is not a very careful way of looking at things. We cannot go into the details of the attempts that have been made to interpret the orthocomplementation in a physical way, and refer to [7–10, 19] for those that are interested in this problem. The problem is also considered in depth in [20–22].

Axiom 3 (Orthocomplementation). *Suppose that we have a physical entity S described by a state property space $(\Sigma, \mathcal{L}, \kappa)$. The lattice $\mathcal{L}$ of properties of the physical entity is orthocomplemented. This means that there exists a function $' : \mathcal{L} \to \mathcal{L}$ such that for $a, b \in \mathcal{L}$ we have:*

$$(a')' \;=\; a \, , \tag{11.9}$$

$$a < b \Longrightarrow b' < a' \, , \tag{11.10}$$

$$a \wedge a' = 0 \;\text{ and }\; a \vee a' = I \, . \tag{11.11}$$

For $\mathcal{P}(\Omega)$ the orthocomplement of a subset is given by the complement of this subset, and for $\mathcal{P}(\mathcal{H})$ the orthocomplement of a closed subspace is given by the subspace orthogonal to this closed subspace.

11.4.5 Axioms 4 and 5: Covering Law and Weak Modularity

The next two axioms are called the covering law and weak modularity. There is no obvious physical interpretation for them. They have been put forward mainly because they are satisfied in the lattice of closed subspaces of a complex Hilbert space.

Axiom 4 (Covering Law). *Suppose that we have a physical entity S described by a state property space $(\Sigma, \mathcal{L}, \kappa)$. The lattice $\mathcal{L}$ of properties of the physical entity satisfies the covering law. This means that for $a, x \in \mathcal{L}$ and $p \in \Sigma$, we have*

$$a < x < a \vee p \Longrightarrow x = a \text{ or } x = a \vee p \,. \tag{11.12}$$

Axiom 5 (Weak Modularity). *Suppose that we have a physical entity S described by a state property space $(\Sigma, \mathcal{L}, \kappa)$. The orthocomplemented lattice $\mathcal{L}$ of properties of the physical entity is weakly modular. This means that for $a, b \in \mathcal{L}$, we have*

$$a < b \Longrightarrow (b \wedge a') \vee a = b \,. \tag{11.13}$$

It can be shown that both axioms, the covering law and weak modularity, are satisfied for the two examples $\mathcal{P}(\Omega)$ and $\mathcal{P}(\mathcal{H})$ [6, 7].

11.4.6 Axiom 6: Plane Transitivity

The first five axioms are modelled on Piron's original representation theorem [6]. The sixth axiom that brings us directly to the structure of one of the three standard Hilbert spaces is much more recent [17].

Axiom 6 (Plane Transitivity). *Suppose that we have a physical entity S described by a state property space $(\Sigma, \mathcal{L}, \kappa)$. The orthocomplemented lattice $\mathcal{L}$ of properties of the physical entity is plane transitive. This means that for all atoms $s, t \in \mathcal{L}$, there exist two distinct atoms s_1, s_2 and a symmetry f such that $f|_{[0, s_1 \vee s_2]}$ is the identity and $f(s) = t$.*

Both classical entities and quantum entities can be described by a state property space where the set of properties is a complete atomistic orthocomplemented lattice that satisfies the covering law, and is weakly modular and plane transitive. Now we have to consider the converse, i.e., how this structure leads us to classical physics and to quantum physics.

11.5 The Representation Theorem

This section is more technical than the above, because we want to show in some detail how the standard quantum mechanical structure emerges from the simple operational structure of a state property space that satisfies the six axioms. Above all, we want to make it clear how the classical and pure quantum parts of the general structure appear. Therefore, we first show how the classical and nonclassical parts can be extracted from the general structure, and then we show how the nonclassical parts can be represented by generalized Hilbert spaces, if they are finite-dimensional, and by one of the three standard Hilbert spaces if they are infinite-dimensional. Since both examples $\mathcal{P}(\Omega)$ and $\mathcal{P}(\mathcal{H})$ satisfy the six axioms, it is clear that a theory where the six axioms are satisfied can give rise to a classical theory, as well as to a quantum theory, but in general gives rise to a mixture of both, in the sense of a quantum theory with superselection rules.

11.5.1 The Classical Part

It is possible to filter out the classical part by introducing the notions of classical property and classical state. We introduce a classical property $a \in \mathcal{L}$ as a property for which, for each state $p \in \Sigma$ of the physical entity, this property a is actual or its orthocomplement property a' is actual. The idea is that a property $a \in \mathcal{L}$ is classical if no indeterminism exists for any test α testing this property. This in turn means that, for each state $p \in \Sigma$, α gives the outcome 'yes' with certainty or α gives the outcome 'no' with certainty when tested.

Definition 4 (Classical Property). *Suppose that $(\Sigma, \mathcal{L}, \kappa)$ is the state property space representing a physical entity S, satisfying Axioms 1, 2 and 3. We say that a property $a \in \mathcal{L}$ is a classical property if, for all $p \in \Sigma$, we have*

$$p \in \kappa(a) \text{ or } p \in \kappa(a') \; . \tag{11.14}$$

The set of all classical properties is denoted by $\mathcal{C}$.

Again considering our two examples, it is easy to see that for the quantum case, hence for $\mathcal{L} = \mathcal{P}(\mathcal{H})$, we have no nontrivial classical properties. Indeed, for any closed subspace $A \in \mathcal{H}$, different from 0 and $\mathcal{H}$, we have rays of $\mathcal{H}$ that are neither contained in A nor contained in A'. These are exactly the rays that correspond to a superposition of states in A and states in A'. It is due to the superposition principle in standard quantum mechanics that the only classical properties of a quantum entity are the trivial ones, represented by 0 and $\mathcal{H}$. It is also easy to see that, for the case of a classical entity described by $\mathcal{P}(\Omega)$, all properties are classical properties. Indeed, consider an arbitrary property $A \in \mathcal{P}(\Omega)$. Then for any singleton $\{p\} \in \Sigma$ representing a state

of the classical entity, we have $\{p\} \subset A$ or $\{p\} \subset A'$, since A' is the set-theoretical complement of A.

Next we introduce the idea of a classical state in the following way. For each state $p \in \Sigma$ of the entity, we consider the set of all classical properties that are actual when the entity is in this state. The infimum of this set of classical properties is a property that is also actual, and that is the greatest property that makes all these classical properties actual when actual itself. Hence it perfectly plays the role of a classical state corresponding to p, namely, a state in which only the classical properties (classical part) of the entity is considered. In an obvious way, we introduce the classical Cartan map as the map that sends each classical property to the set of classical states that make this property actual.

Definition 5 (Classical State). *Suppose that $(\Sigma, \mathcal{L}, \kappa)$ is the state property space of a physical entity S satisfying Axioms 1, 2 and 3. For $p \in \Sigma$, we introduce*

$$\omega(p) = \bigwedge_{p \in \kappa(a), a \in \mathcal{C}} a \,, \tag{11.15}$$

$$\kappa_{\mathrm{c}}(a) = \{\omega(p) |\ p \in \kappa(a)\} \,, \tag{11.16}$$

and call $\omega(p)$ the classical state of the physical entity whenever it is in a state $p \in \Sigma$, and κ_{c} the classical Cartan map. The set of all classical states will be denoted by Ω.

We have now introduced all that is needed to define the classical state property space of the entity under consideration.

Definition 6 (Classical State Property Space). *Suppose that $(\Sigma, \mathcal{L}, \kappa)$ is the state property space of a physical entity satisfying Axioms 1, 2 and 3. The classical state property space corresponding to $(\Sigma, \mathcal{L}, \kappa)$ is $(\Omega, \mathcal{C}, \kappa_{\mathrm{c}})$.*

Let us look at our two examples. For the quantum case, with $\mathcal{L} = \mathcal{P}(\mathcal{H})$, we have only two classical properties, namely 0 and $\mathcal{H}$. This means that there is only one classical state, namely $\mathcal{H}$. It is the classical state that corresponds to considering the quantum entity under study and the state does not specify anything more than that. For the classical case, every state is a classical state.

It can be proven that $\kappa_{\mathrm{c}} : \mathcal{C} \to \mathcal{P}(\Omega)$ is an isomorphism [8,10]. This means that if we filter out the classical part and limit the description of our general physical entity to its classical properties and classical states, the description becomes a standard classical physical description.

11.5.2 The Nonclassical Parts

Now that we have identified the classical parts, let us filter out the nonclassical part. The idea is that we now consider the physical entity to be in a specific classical state ω, and then penetrate further into the leftover nonclassical aspects of this entity.

Definition 7 (Nonclassical Part). *Suppose that* $(\Sigma, \mathcal{L}, \kappa)$ *is the state property space of a physical entity satisfying Axioms 1, 2 and 3. For* $\omega \in \Omega$*, we introduce*

$$\mathcal{L}_\omega = \{a | a < \omega, a \in \mathcal{L}\} \ , \tag{11.17}$$

$$\Sigma_\omega = \{p | p \in \kappa(\omega), p \in \Sigma\} \ , \tag{11.18}$$

$$\kappa_\omega(a) = \kappa(a) \text{ for } a \in \mathcal{L}_\omega \ , \tag{11.19}$$

and we call $(\Sigma_\omega, \mathcal{L}_\omega, \kappa_\omega)$ *the nonclassical components of* $(\Sigma, \mathcal{L}, \kappa)$.

For the quantum case $\mathcal{L} = \mathcal{P}(\mathcal{H})$, we have only one classical state $\mathcal{H}$, and obviously $\mathcal{L}_\mathcal{H} = \mathcal{L}$. Similarly we have $\Sigma_\mathcal{H} = \Sigma$. This means that the only nonclassical component is $(\Sigma, \mathcal{L}, \kappa)$ itself. For the classical case, since all properties are classical properties and all states are classical states, we have $\mathcal{L}_\omega = \{0, \omega\}$, which is the trivial lattice, containing only its minimal and maximal element, and $\Sigma_\omega = \{\omega\}$. This means that the nonclassical components are all trivial.

For the general situation of a physical entity described by $(\Sigma, \mathcal{L}, \kappa)$, it can be shown that $\mathcal{L}_\omega$ contains no classical properties with respect to Σ_ω except 0 and ω, the minimal and maximal elements of $\mathcal{L}_\omega$, and that if $(\Sigma, \mathcal{L}, \kappa)$ satisfies Axioms 1–6 then $(\Sigma_\omega, \mathcal{L}_\omega, \kappa_\omega)$ $\forall \omega \in \Omega$ also satisfy Axioms 1–6 (see [8,10,17]).

We note that, if Axioms 1, 2 and 3 are satisfied, we can identify a state $p \in \Sigma$ with the element of the lattice of properties $\mathcal{L}$ given by

$$s(p) = \bigwedge_{p \in \kappa(a), a \in \mathcal{L}} a \ , \tag{11.20}$$

which is an atom of $\mathcal{L}$. More precisely, it is not difficult to verify that, under the assumption of Axioms 1 and 2, $s : \Sigma \to \Sigma_\mathcal{L}$ is a well-defined mapping that is one-to-one and onto, $\Sigma_\mathcal{L}$ being the collection of all atoms in $\mathcal{L}$. Moreover, $p \in \kappa(a)$ iff $s(p) < a$. We can call $s(p)$ the property state corresponding to p and define

$$\Sigma' = \{s(p) |\ p \in \Sigma\} \tag{11.21}$$

as the set of state properties. It is easy to verify that, if we introduce

$$\kappa' : \mathcal{L} \longrightarrow \mathcal{P}(\Sigma') \ , \tag{11.22}$$

where

$$\kappa'(a) = \{s(p) |\ p \in \kappa(a)\} \ , \tag{11.23}$$

then

$$(\Sigma', \mathcal{L}, \kappa') \cong (\Sigma, \mathcal{L}, \kappa) \ , \tag{11.24}$$

when Axioms 1, 2 and 3 are satisfied.

To see in more detail how the classical and nonclassical parts are structured within the lattice $\mathcal{L}$, we make use of this isomorphism and introduce the direct union of a set of complete, atomistic orthocomplemented lattices, making use of this identification.

Definition 8 (Direct Union). *Consider a set $\{\mathcal{L}_\omega | \omega \in \Omega\}$ of complete, atomistic orthocomplemented lattices. The direct union $\bigcirc\!\!\!\!\vee_{\omega\in\Omega}\mathcal{L}_\omega$ of these lattices consists of the sequences $a = (a_\omega)_\omega$, such that*

$$(a_\omega)_\omega < (b_\omega)_\omega \iff a_\omega < b_\omega \ \forall \omega \in \Omega \ , \tag{11.25}$$

$$(a_\omega)_\omega \wedge (b_\omega)_\omega = (a_\omega \wedge b_\omega)_\omega \ , \tag{11.26}$$

$$(a_\omega)_\omega \vee (b_\omega)_\omega = (a_\omega \vee b_\omega)_\omega \ , \tag{11.27}$$

$$(a_\omega)'_\omega = (a'_\omega)_\omega \ . \tag{11.28}$$

The atoms of $\bigcirc\!\!\!\!\vee_{\omega\in\Omega}\mathcal{L}_\omega$ are of the form $(a_\omega)_\omega$, where $a_{\omega_1} = p$ for some ω_1 and $p \in \Sigma_{\omega_1}$, and $a_\omega = 0$ for $\omega \neq \omega_1$.

It can be proven that if $\mathcal{L}_\omega$ are complete, atomistic, orthocomplemented lattices, then $\bigcirc\!\!\!\!\vee_{\omega\in\Omega}\mathcal{L}_\omega$ is also a complete, atomistic, orthocomplemented lattice (see [8, 10]). The structure of a direct union of complete, atomistic, orthocomplemented lattices makes it possible to define the direct union of state property spaces when Axioms 1, 2 and 3 are satisfied.

Definition 9 (Direct Union of State Property Spaces). *Let $(\Sigma_\omega, \mathcal{L}_\omega, \kappa_\omega)$ be a set of state property spaces, where $\mathcal{L}_\omega$ are complete, atomistic, orthocomplemented lattices and for each ω we have that Σ_ω is the set of atoms of $\mathcal{L}_\omega$. The direct union $\bigcirc\!\!\!\!\vee_\omega(\Sigma_\omega, \mathcal{L}_\omega, \kappa_\omega)$ of these state property spaces is the state property space $(\cup_\omega\Sigma_\omega, \bigcirc\!\!\!\!\vee_\omega\mathcal{L}_\omega, \bigcirc\!\!\!\!\vee_\omega\kappa_\omega)$, where $\cup_\omega\Sigma_\omega$ is the disjoint union of the sets Σ_ω, $\bigcirc\!\!\!\!\vee_\omega\mathcal{L}_\omega$ is the direct union of the lattices $\mathcal{L}_\omega$, and*

$$\bigcirc\!\!\!\!\vee_\omega\kappa_\omega\big((a_\omega)_\omega\big) = \cup_\omega\kappa_\omega(a_\omega) \ . \tag{11.29}$$

The first part of a fundamental representation theorem can now be stated. For this part, Axioms 1, 2 and 3 suffice.

Theorem 1 (Representation Theorem: Part 1). *We consider a physical entity S described by its state property space $(\Sigma, \mathcal{L}, \kappa)$. Suppose that Axioms 1, 2 and 3 are satisfied. Then*

$$(\Sigma, \mathcal{L}, \kappa) \cong \bigcirc\!\!\!\!\vee_{\omega\in\Omega}(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega) \ , \tag{11.30}$$

where Ω is the set of classical states of $(\Sigma, \mathcal{L}, \kappa)$ (see Definition 5), Σ'_ω is the set of state properties, κ'_ω the corresponding Cartan map [see (11.21) and (11.23)], and $\mathcal{L}_\omega$ the lattice of properties (see Definition 7) of the nonclassical component $(\Sigma_\omega, \mathcal{L}_\omega, \kappa_\omega)$. If Axioms 4, 5 and 6 are satisfied for $(\Sigma, \mathcal{L}, \kappa)$, then they are also satisfied for $(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega)$, for all $\omega \in \Omega$.

Proof. See [8, 10, 17].

From the previous section, it follows that, if Axioms 1–6 are satisfied, we can write the state property space $(\Sigma, \mathcal{L}, \kappa)$ of the physical entity under study as the direct union $\bigcirc\!\!\!\!\vee_{\omega\in\Omega}(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega)$ over the classical state space Ω of its nonclassical components $(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega)$, and that each of these nonclassical components also satisfies Axioms 1–6. Additionally, for each of the nonclassical components $(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega)$, no classical properties except 0 and ω exist. It is for the nonclassical components that a further representation theorem can be proven such that a vector space structure emerges for each of the nonclassical components. To do this, we rely on the original representation theorem proved by Piron in [6] and on the more recent results proved in [17].

Theorem 2 (Representation Theorem: Part 2). *Consider the same situation as in Theorem 1, but with Axioms 4, 5 and 6 satisfied as well. For each nonclassical component $(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega)$, of which the lattice $\mathcal{L}_\omega$ has at least four orthogonal states,*[7] *there exists a vector space V_ω over a division ring K_ω, with an involution of K_ω, which means a function*

$$^* : K_\omega \longrightarrow K_\omega \ , \tag{11.31}$$

such that, for $k, l \in K_\omega$, we have

$$(k^*)^* = k \ , \tag{11.32}$$

$$(k \cdot l)^* = l^* \cdot k^* \ , \tag{11.33}$$

and a Hermitian product on V_ω, which means a function

$$\langle \ , \ \rangle : V_\omega \times V_\omega \longrightarrow K_\omega \ , \tag{11.34}$$

such that, for $x, y, z \in V_\omega$ and $k \in K_\omega$, we have

$$\langle x + ky, z\rangle = \langle x, z\rangle + k\langle x, y\rangle \ , \tag{11.35}$$

$$\langle x, y\rangle^* = \langle y, x\rangle \ , \tag{11.36}$$

$$\langle x, x\rangle = 0 \Longleftrightarrow x = 0 \ , \tag{11.37}$$

and such that, for $M \subset V_\omega$, we have

$$M^\perp + (M^\perp)^\perp = V_\omega \ , \tag{11.38}$$

[7] Two states $p, q \in \Sigma_\omega$ are orthogonal if there exists $a \in \mathcal{L}_\omega$ such that $p < a$ and $q < a'$.

where $M^{\perp} = \{y | y \in V_\omega, \langle y, x\rangle = 0, \forall x \in M\}$. *Such a vector space is called a generalized Hilbert space or an orthomodular vector space. Furthermore,*

$$(\Sigma'_\omega, \mathcal{L}_\omega, \kappa'_\omega) \cong \left(\mathcal{R}(V), \mathcal{L}(V), \nu\right) , \tag{11.39}$$

where $\mathcal{R}(V)$ *is the set of rays of* V, $\mathcal{L}(V)$ *is the set of biorthogonally closed subspaces of* V *(subspaces that are equal to their biorthogonal), and* ν *maps each such biorthogonal subspace to the set of rays that are contained in it. Moreover, if a classical component is infinite-dimensional, which means that it contains an infinite sequence of orthogonal atoms, the generalized Hilbert space is isomorphic to a real, complex or quaternionic Hilbert space.*

Proof. See [6, 7, 17].

11.6 Failing Axioms: Separated Entities

In the last section, we explained in detail the representation theorem that makes it possible to move, by means of 6 axioms, from the completely operational structure of a state property space to standard quantum mechanics with superselection rules. Since we have only presented the results here and not the proofs, we should mention that three long, hard mathematical proofs underpin this result. First there is the Piron representation theorem, which brings us by means of 5 axioms to a generalized Hilbert space (with superselection rules) [6,7]. This representation theorem makes use of the fundamental theorem of projective geometry [23], one of the long and hard standard mathematical proofs. Second there is Solèr's theorem which uses Axiom 6 to bring us to one of the standard Hilbert spaces for the infinite-dimensional nonclassical components [16]. This is also a long, hard mathematical theorem, and not yet standard, because it is very recent, but certainly the important steps of it will become standard. Last but not least, we have not yet mentioned quantum probability. But since we have now arrived at the standard Hilbert spaces, we can use Gleason's theorem to derive the standard quantum mechanical transition probability in a unique way. This means that to get from our state property space with the six axioms to standard quantum mechanics with superselection rules, long, hard mathematical proofs are needed. It is a quite powerful situation to have been able to concentrate all this wealth of mathematical structure inside the transparent operational structure of a state property space and 6 axioms.

One of the aspects of this power is that we can investigate the status of these axioms. More specifically we can look at situations that have caused deep problems to be described by standard quantum mechanics and investigate which of the axioms are at the origin of these problems. This is exactly what we have done in the past decades and we have been able to prove that Axioms 4 and 5 are at the origin of two essential shortcomings of standard

quantum mechanics: its incapacity to describe separated quantum entities and its incapacity to describe a continuous transition from quantum to classical. Within the approach to state property spaces in which Axioms 4 and 5 are relaxed, the description of separated quantum entities is possible and it is also possible to describe a continuous transition from quantum to classical. We will now explain in somewhat more detail what really happens for the description of separated physical entities. We analyze how Axioms 4 and 5 make it impossible to describe a continuous transition from quantum to classical in the next section, because for this analysis we need the explanation of quantum probabilities.

11.6.1 The Impossibility of Describing Separated Entities

Let us first explain what is meant by separated physical entities. We consider a physical entity S that consists of two physical entities S_1 and S_2. The definition of 'separated' used in [8,9] is as follows. Suppose that we consider two experiments e_1 and e_2 that can be performed on the entity S_1 and on the entity S_2, respectively, such that the joint experiments $e_1 \times e_2$ can be performed on the joint entity S consisting of S_1 and S_2. We say that experiments e_1 and e_2 are separated experiments whenever, for an arbitrary state p of S, we have that (x_1, x_2) is a possible outcome for experiment $e_1 \times e_2$ if and only if x_1 is a possible outcome for e_1 and x_2 is a possible outcome for e_2. We say that S_1 and S_2 are separated entities if and only if all the experiments e_1 on S_1 are separated from the experiments e_2 on S_2.

Let us note that S_1 and S_2 being separated does not mean that there is no interaction between S_1 and S_2. Most entities in the macroscopic world are separated entities. Let us consider some examples to make this clear. The earth and the moon, for example, are separated entities. Indeed, consider any experiment e_1 that can be performed on the physical entity earth (e.g., measuring its position) and any experiment e_2 that can be performed on the physical entity moon (for example measuring its velocity). The joint experiment $e_1 \times e_2$ consists in performing e_1 and e_2 together on the joint entity of earth and moon (measuring the position of the earth and the velocity of the moon at once). Obviously the requirement of separation is satisfied. The pair (x_1, x_2) (position of the earth and velocity of the moon) is a possible outcome for $e_1 \times e_2$ if and only if x_1 (position of the earth) is a possible outcome of e_1 and x_2 (velocity of the moon) is a possible outcome of e_2. This is what we mean when we say that the earth has position x_1 and the moon velocity x_2 at once. Clearly this is independent of whether there is an interaction, the gravitational interaction in this case, between the earth and the moon.

It is not easy to find an example of two physical entities that are not separated in the macroscopic world, because nonseparated entities are usually described as one entity and not as two. Elsewhere we have given examples of nonseparated macroscopic entities [24–26]. The example of communicating

vessels of water is a good example to give an intuitive idea of what nonseparation means. Consider two vessels V_1 and V_2, each containing 10 liters of water. The vessels communicate by a tube, which means that they form a communicating set of vessels. The tube also contains some water, but this does not play any role in what we want to show. Experiment e_1 consists in taking water out of vessel V_1 by a siphon, and measuring the amount of water that comes out. We give the outcome x_1 if the amount of water coming out is greater than 10 liters. Experiment e_2 consists in doing exactly the same with vessel V_2. We give outcome x_2 to e_2 if the amount of water coming out is greater than 10 liters. The joint experiment $e_1 \times e_2$ consists in performing e_1 and e_2 together on the joint entity of the two communicating vessels of water. Because of the connection, and the physical principles that govern communicating vessels, for e_1 and for e_2 performed alone we find 20 liters of water coming out. This means that x_1 is a possible (even certain) outcome for e_1 and x_2 is a possible (also certain) outcome for e_2. If we perform the joint experiment $e_1 \times e_2$, the following happens. If more than 10 liters comes out of vessel V_1, less than 10 liters will come out of vessel V_2 and if more than 10 liters comes out of vessel V_2, less than 10 liters will come out of vessel V_1. This means that (x_1, x_2) is not a possible outcome for the joint experiment $e_1 \times e_2$. Hence e_1 and e_2 are nonseparated experiments and as a consequence V_1 and V_2 are nonseparated entities.

The nonseparated entities that we find in the macroscopic world are entities that are very similar to the communicating vessels of water. There must be an ontological connection between the two entities, and that is also the reason why the joint entity will usually be treated as a single entity. A connection through dynamic interaction, as is the case between the earth and the moon, interacting by gravitation, leaves the entities separated.

For quantum entities, it can be shown that only when the joint entity of two quantum entities contains entangled states are the entities nonseparated quantum entities. It can be proven [24–26] that experiments are separated if and only if they do not violate Bell's inequalities. All this has been explored and investigated in many ways, and several papers have been published on the matter [24–28]. Interesting consequences for the Einstein–Podolsky–Rosen paradox and the violation of Bell's inequalities have been investigated [29,30].

11.6.2 The Separated Quantum Entities Theorem

We are now ready to state the theorem about the impossibility for standard quantum mechanics to describe separated quantum entities [8,9]. The demand of separation explained in the last section can easily be transferred to the state property spaces just by demanding that the yes/no experiments that test properties of one of the entities be separated from yes/no experiments testing properties of the other entity.

Theorem 3 (Separated Quantum Entities Theorem). *Suppose that S is a physical entity consisting of two separated physical entities S_1 and S_2.*

Let us suppose that Axioms 1, 2 and 3 are satisfied and call $(\Sigma, \mathcal{L}, \kappa)$ *the state property space describing* S*, and* $(\Sigma_1, \mathcal{L}_1, \kappa_1)$ *and* $(\Sigma_2, \mathcal{L}_2, \kappa_2)$ *the state property spaces describing* S_1 *and* S_2*.*

- *If Axiom 4 is satisfied, namely the covering law, then one of the two entities* S_1 *or* S_2 *is a classical entity, in the sense that one of the two state property spaces* $(\Sigma_1, \mathcal{L}_1, \kappa_1)$ *or* $(\Sigma_2, \mathcal{L}_2, \kappa_2)$ *contains only classical states and classical properties.*
- *If Axiom 5 is satisfied, namely weak modularity, then one of the two entities* S_1 *or* S_2 *is a classical entity, in the sense that one of the two state property spaces* $(\Sigma_1, \mathcal{L}_1, \kappa_1)$ *or* $(\Sigma_2, \mathcal{L}_2, \kappa_2)$ *contains only classical states and classical properties.*

Proof. See [8,9].

The theorem proves that two separated quantum entities cannot be described by standard quantum mechanics. A classical entity that is separated from a quantum entity and two separated classical entities do not cause any problem, but two separated quantum entities need a structure in which neither the covering law nor weak modularity is satisfied.

One possible way out is that there might not exist separated quantum entities in nature. This would mean that all quantum entities are entangled in some way or other. If this is true, perhaps the standard formalism could be saved. Let us note, however, that even standard quantum mechanics presupposes the existence of separated quantum entities. Indeed, if we describe one quantum entity by means of the standard formalism, we take one Hilbert space to represent the states of this entity. In this sense we suppose the rest of the universe to be separated from this one quantum entity. If not, we would have to modify the description and consider two Hilbert spaces, one for the entity and one for the rest of the universe, and the states would be entangled states of the states of the entity and the states of the rest of the universe. But this would mean that the one quantum entity that we considered could never be in a well-defined state. It would mean that the only possibility that remains is to describe the whole universe at once by using one huge Hilbert space.

It goes without saying that such an approach will lead to many other problems. For example, if this one Hilbert space has to describe the whole universe, will it also contain itself, as a description, because as a description, a human activity, it is part of the whole universe. Another more down to earth problem is that, in this one Hilbert space of the whole universe, all classical macroscopic entities also have to be described. But classical entities are not described by a Hilbert space, as we have seen in Sect. 11.5. If the hypothesis that we can only describe the whole universe at once is correct, one would expect the theory that does deliver such a description to be the direct union structure of different Hilbert spaces. But if this is the case, we are already using a more general theory than standard quantum mechanics.

So we may as well use the still slightly more general theory, where Axioms 4 and 5 are not satisfied, and make the description of separated quantum entities possible.

All this convinces us that the inability of standard quantum mechanics to describe separated quantum entities is really a shortcoming of the mathematical formalism used by standard quantum mechanics, and more notably of the vector space structure of the Hilbert space used in standard quantum mechanics.

11.7 An Explanation for Quantum Probability

The axiomatics that we have outlined in the foregoing sections still lacks a description of a very fundamental notion: probability. We may choose to derive the probabilistic features of quantum mechanics by means of Gleason's theorem, but this is really a great detour. It is our view that probability should be introduced on a much more profound level, because it is an operationally well-defined aspect of repeated measurements. The reason that the axiomatics were built in this way, neglecting probability, is largely historical. As such, we felt an investigation into the probabilistic aspects of our approach was called for. What we have been able to show for quantum probability can best be illustrated by a very simple macroscopic example which, as we shall see, also constitutes a model for the spin of a spin 1/2 quantum entity.

11.7.1 The Sphere Model

The example that we want to introduce consists of a physical entity constituted by a point particle P that can move on the surface of a sphere, denoted $\mathcal{S}$, with center O and radius 1. The unit vector v giving the location of the particle on $\mathcal{S}$ represents the state p_v of the particle (see Fig. 11.1a). Hence, the collection of all possible states of the sphere model, as we shall call our model, is given by $\Sigma = \{p_v |\ v \in \mathcal{S}\}$. We introduce the following yes/no experiments. For each point $u \in \mathcal{S}$, we introduce the experiment α_u. We consider the diametrically opposite point $-u$ and install an elastic band of length 2, such that it is fixed with one of its endpoints at u and the other endpoint at $-u$. Once the elastic is in place, the particle P falls from its original place v orthogonally onto the elastic and sticks to it (Fig. 11.1b). The elastic then breaks and the particle P, attached to one of the two pieces of the elastic (Fig. 11.1c), moves to one of the two endpoints u or $-u$ (Fig. 11.1d). Depending on whether the particle P arrives at u (as in Fig. 11.1) or at $-u$, we give the outcome 'yes' or 'no' to α_u. The state p_v is changed by the experiment α_u into one of the two states p_u or p_{-u}.

We make the hypothesis that the elastic band breaks uniformly, which means that the probability that a particle in state p_v arrives at u is given by the length of L_1 (which is $1 + \cos\theta$) divided by the total length of the

elastic (which is 2). The probability that a particle in state p_v arrives at $-u$ is given by the length of L_2 (which is $1 - \cos\theta$) divided by the total length of the elastic. If we denote these probabilities by $P(\alpha_u, p_v)$ and $P(\alpha_{-u}, p_v)$, respectively, we have

$$P(\alpha_u, p_v) = \frac{1 + \cos\theta}{2} = \cos^2 \frac{\theta}{2} \,, \tag{11.40}$$

$$P(\alpha_{-u}, p_v) = \frac{1 - \cos\theta}{2} = \sin^2 \frac{\theta}{2} \,. \tag{11.41}$$

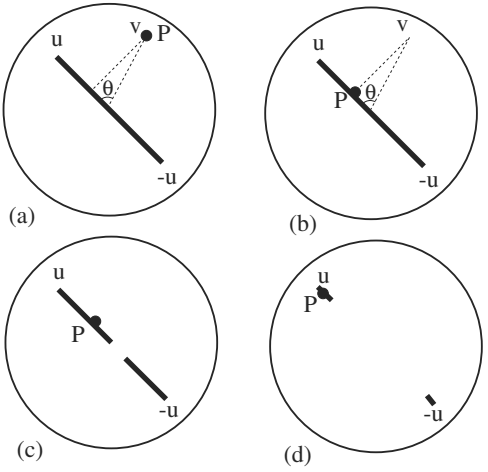

Fig. 11.1. Sphere model. (**a**) The physical entity P is in state p_v at the point v, and the elastic corresponding to the experiment α_u is installed between the two diametrically opposed points u and $-u$. (**b**) The particle P falls orthogonally onto the elastic and sticks to it. (**c**) The elastic breaks and the particle P is pulled towards the point u, so that (**d**) it arrives at the point u, and the experiment α_u gets the outcome 'yes'

Figure 11.2 shows the experimental process connected to α_u in the plane where it takes place, and we can easily calculate the probabilities corresponding to the two possible outcomes. In order to do so, we note that the particle P arrives at u when the elastic breaks at a point of the interval L_1, and arrives at $-u$ when it breaks at a point of the interval L_2 (see Fig. 11.2).

We can easily show that the sphere model is an entity for which the description is isomorphic to the quantum description of the spin of a spin 1/2 particle, and as such delivers a model for this. This means that we can describe this macroscopic entity using the ordinary quantum formalism with a two-dimensional complex Hilbert space as the carrier for the set of states of the entity.

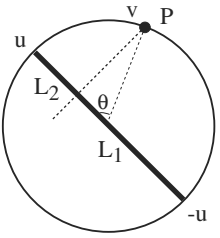

Fig. 11.2. A representation of the experimental process in the plane where it takes place. The elastic of length 2, corresponding to the experiment α_u, is installed between u and $-u$. The probability $P(\alpha_u, p_v)$ that the particle P ends up at point u is given by the length of the piece of elastic L_1 divided by the total length of the elastic. The probability $P(\alpha_{-u}, p_v)$, that the particle P ends up at point $-u$ is given by the length of the piece of elastic L_2 divided by the total length of the elastic

We note that the sphere model is an elaboration of the well known Bloch or Poincaré model for the spin of a spin 1/2 particle, including also a modeling of the spin measurements. The sphere model as a model for an arbitrary quantum system described by a two-dimensional Hilbert space was presented in [31–33]. It is possible to prove that, for an arbitrary quantum entity, one can construct a model like that of the sphere model [34–37]. The explanation of the quantum structure that is given in the sphere model can thus also be used for general quantum entities. We have called this explanation the 'hidden measurement approach', hidden measurements referring to the fact that, for a real measurement, there is a lack of knowledge about the measurement process in this approach. For the sphere model, for example, this lack of knowledge is the lack of knowledge about where the elastic will break during a measurement process.

11.7.2 What are Quantum Structures and Why Do They Appear in Nature?

The explanation for the quantum probabilities that we have put forward within the hidden measurement approach makes it possible to identify the reason why quantum structures appear in a natural way in nature.

The original development of probability theory aimed at a formalization of the description of the probabilities that appear as a consequence of a lack of knowledge. The probability structure appearing in situations of lack of knowledge was axiomatized by Kolmogorov and such a probability model is now called Kolmogorovian. Since the quantum probability model is not Kolmogorovian, it has now generally been accepted that the quantum probabilities are not associated with a lack of knowledge. Sometimes this conclusion is formulated by stating that the quantum probabilities are ontological probabilities, as if they were present in reality itself. In the hidden measurement

approach, we show that the quantum probabilities can also be explained as being due to a lack of knowledge, and we prove that what distinguishes quantum probabilities from classical Kolmogorovian probabilities is the nature of this lack of knowledge. Let us go back to the sphere model to illustrate what we mean.

If we return to our sphere model (Figs. 11.1 and 11.2) and look for the origin of the probabilities as they appear in this example, we observe that the probability is entirely due to a lack of knowledge about the measurement process, namely the lack of knowledge as to where exactly the elastic breaks during a measurement. More specifically, we can identify two main aspects of the experiment α_u as it appears in the sphere model.

1. The experiment α_u effects a real change on the state p_v of the point P. Indeed, the state p_v is changed into one of the states p_u or p_{-u} by the experiment α_u.
2. The probabilities appearing are due to a lack of knowledge about a deeper reality of the individual measurement process itself, namely, where the elastic breaks.

These two effects give rise to quantum-like structures, and the lack of knowledge about the deeper reality of the individual measurement process comes from 'hidden measurements' that operate deterministically in this deeper reality [31–33, 39, 41, 42].

One might think that our 'hidden-measurement' approach is in fact a 'hidden-variable' theory. In a certain sense this is true. If our explanation for the quantum structures is the correct one, quantum mechanics is compatible with a deterministic universe at the deepest level. There is no need to introduce the idea of an ontological probability. Why then the generally held conviction that hidden variable theories cannot be used for quantum mechanics? The reason is that those physicists who are interested in trying out hidden variable theories are not at all interested in the kind of theory that we propose here. They want the hidden variables to be hidden variables of the state of the entity under study, so that the probability is associated with a lack of knowledge about the deeper reality of this entity; as we have mentioned already this gives rise to a Kolmogorovian probability theory. This kind of hidden variable relating to states is indeed impossible for quantum mechanics for structural reasons, with the exception of course of the de Broglie–Bohm theory: there, in addition to the hidden state variables, a new spooky entity of 'quantum potential' is introduced in order to express the action of the measurement as a change in the hidden state variables.

If one wants to interpret our hidden measurements as hidden variables, then they are hidden variables of the measuring apparatus and not of the entity under study. In this sense they are highly contextual, since each experiment introduces a different set of hidden variables. They differ from the variables of a classical hidden variable theory, because they do not provide an 'additional deeper' description of the reality of the physical entity. Their

presence as variables of the experimental apparatus has a well-defined philosophical meaning, and expresses the fact that we, human beings, want to construct a model of reality independent of our experience of this reality. The reason is that we look for 'properties' or 'relations between properties', and these are defined by our ability to make predictions independent of our experience. We want to model the structure of the world independently of our observing and experimenting with this world. Since we do not control these variables in the experimental apparatus, we do not allow them in our model of reality, and the probability introduced by them cannot be eliminated from a predictive theoretical model. In the macroscopic world, because of the availability of many experiments with negligible fluctuations, we find an 'almost' deterministic model. For a detailed study of other aspects of the hidden measurement approach we refer to [30–47].

11.7.3 A Transition from Quantum to Classical

It is a great mystery how the macroscopic classical world around us has emerged from the microscopic quantum world. To our knowledge, no satisfactory model or explanation has been given for this. What is certain is that somewhere and somehow a transition has to have happened. Taking into account the sphere model, we can propose a simple model for such a transition: we introduce a parameter ϵ that parameterizes the amount of fluctuation that is present on the interaction between measuring apparatus and physical entity. We have called the extended sphere model that appears in this way the ϵ-model.

More specifically we introduce two real parameters $\epsilon \in [0,1]$, and $d \in [-1+\epsilon, 1-\epsilon]$, and consider the experiment $\alpha^{\epsilon}_{u,d}$ that consists in the particle P falling from its original place v orthogonally onto the line between u and $-u$, and arriving at a point coordinated by the real number $v \cdot u$. The hypothesis is that the rubber band never breaks outside the interval $[d-\epsilon, d+\epsilon]$. In the interval $[d-\epsilon, d+\epsilon]$ we consider a uniformly distributed random variable λ, and the experiment proceeds as follows. If $\lambda \in [d-\epsilon, v \cdot u)$, the particle P moves to the point u and the experiment $\alpha^{\epsilon}_{u,d}$ gives outcome 'yes'. If $\lambda \in (v \cdot u, d+\epsilon]$, it moves to the point $-u$, and the experiment $\alpha^{\epsilon}_{u,d}$ gives outcome 'no'. If $\lambda = v \cdot u$, it moves with probability $1/2$ to the point u, and the experiment $\alpha^{\epsilon}_{u,d}$ gives outcome 'yes', and it moves with probability $1/2$ to the point $-u$, and then the experiment $\alpha^{\epsilon}_{u,d}$ gives outcome 'no'. This completes the description of the experiment $\alpha^{\epsilon}_{u,d}$.

We shall now consider different situations labeled by the parameter ϵ.

- $d + \epsilon \leq v \cdot u$. Then $P(\alpha^{\epsilon}_{u,d}, p_v) = 1$ and $P(\alpha^{\epsilon}_{-u,d}, p_v) = 0$.
- $d - \epsilon < v \cdot u < d + \epsilon$. Then

$$P(\alpha^{\epsilon}_{u,d}, p_v) = \frac{1}{2\epsilon}(v \cdot u - d + \epsilon) \ , \tag{11.42}$$

$$P(\alpha^{\epsilon}_{-u,d}, p_v) = \frac{1}{2\epsilon}(d + \epsilon - v \cdot u) \ . \tag{11.43}$$

- $v \cdot u \leq d - \epsilon$. Then $P(\alpha^{\epsilon}_{u,d}, p_v) = 0$ and $P(\alpha^{\epsilon}_{-u,d}, p_v) = 1$.

We have the following situation: there are regions of eigenstates, one region centered around the point u, which we denote $\text{eig}(\alpha^{\epsilon}_{u,d})$, and another one centered around the point $-u$, which we denote $\text{eig}(\alpha^{\epsilon}_{-u,d})$. These regions of eigenstates are spherical sectors of $\mathcal{S}$. Let us denote a closed spherical sector centered around the point $u \in \mathcal{S}$ with angle θ by $\text{sect}(u, \theta)$. We note that, in the classical situation, for $\epsilon = 0$, $\text{eig}(\alpha^{\epsilon}_{u,d})$ and $\text{eig}(\alpha^{\epsilon}_{-u,d})$ are given by open spherical sectors centered around u and $-u$. We denote an open spherical sector centered around u and with angle θ by $\text{sect}^{\text{o}}(u, \theta)$. We call λ^{ϵ}_d the angle of the spherical sector corresponding to $\text{eig}(\alpha^{\epsilon}_{u,d})$ for all u. Hence for $0 \neq \epsilon$, we have $\text{eig}(\alpha^{\epsilon}_{u,d}) = \{p_v |\ v \in \text{sect}(u, \lambda^{\epsilon}_d)\}$, and $\text{eig}(\alpha^{0}_{u,d}) = \{p_v |\ v \in \text{sect}^{\text{o}}(u, \lambda^{0}_d)\}$. We can easily verify that $\text{eig}(\alpha^{\epsilon}_{-u,d})$ is determined by a spherical sector centered around the point $-u$. We call μ^{ϵ}_d the angle of this spherical sector. Hence, for $0 < \epsilon$, we have $\text{eig}(\alpha^{\epsilon}_{-u,d}) = \{p_v |\ v \in \text{sect}(-u, \mu^{\epsilon}_d)\}$ and $\text{eig}(\alpha^{0}_{-u,d}) = \{p_v |\ v \in \text{sect}^{\text{o}}(-u, \mu^{\epsilon}_d)\}$. Let σ^{ϵ}_d denote the angle of superposition states. Then,

$$\cos \lambda^{\epsilon}_d = \epsilon + d \ , \tag{11.44}$$

$$\cos \mu^{\epsilon}_d = \epsilon - d \ , \tag{11.45}$$

$$\sigma^{\epsilon}_d = \pi - \lambda^{\epsilon}_d - \mu^{\epsilon}_d \ , \tag{11.46}$$

$$\lambda^{\epsilon}_{-d} = \mu^{\epsilon}_d \text{ and } \sigma^{\epsilon}_{-d} = \sigma^{\epsilon}_d \ . \tag{11.47}$$

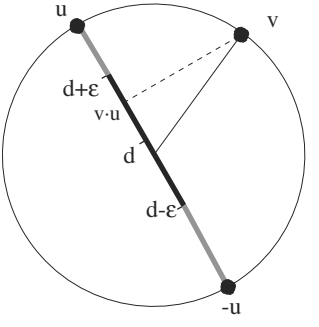

Fig. 11.3. Representation of the experiment $\alpha^{\epsilon}_{u,d}$. We have chosen a case where $d = 0.2$, and $\epsilon = 1/2$. If we consider the elastic sphere example realizing this situation, then the elastic breaks uniformly inside the interval $[d - \epsilon, d + \epsilon]$, and is unbreakable outside this interval, at the points of the set $[-1, d - \epsilon] \cup [d + \epsilon, 1]$

The Quantum Situation ($\epsilon = 1$)

For $\epsilon = 1$, we always have $d = 0$, and the ϵ-model becomes the sphere model of the previous section, hence a quantum model for the spin of a spin 1/2 quantum entity. For the eigenstate sets, we find

$$\text{eig}(\alpha^1_{u,0}) = \{p_v| \ +1 \leq v \cdot u\} = \{p_u\} \ , \tag{11.48}$$

$$\text{eig}(\alpha^1_{-u,0}) = \{p_v| \ v \cdot u \leq -1\} = \{p_{-u}\} \ , \tag{11.49}$$

which shows that the eigenstates are the states p_u and p_{-u}, and all the other states are superposition states.

The Classical Situation ($\epsilon = 0$)

The classical situation is the situation without fluctuations. If $\epsilon = 0$, then d can take any value in the interval $[-1, +1]$, and we have

$$\text{eig}(\alpha^0_{u,d}) = \{p_v| \ d < v \cdot u\} \ , \tag{11.50}$$

$$\text{eig}(\alpha^0_{-u,d}) = \{p_v| \ v \cdot u < d\} \ , \tag{11.51}$$

which shows that for the classical situation, the only superposition states are the states p_v such that $v \cdot u = d$, and all the other states are eigenstates.

A continuous transition from a pure quantum entity to a pure classical entity is possible if we consider the ϵ-model and the transition from $\epsilon = 1$ to $\epsilon = 0$. For values of ϵ such that $0 < \epsilon < 1$, we have an entity that is neither quantum nor classical, but 'intermediate'.

As mentioned earlier, we can prove that the intermediate situations do not satisfy Axioms 4 and 5. Let us put forward the two theorems that we have proven in this respect and indicate where the proofs can be found.

Theorem 4. *If Axiom 4 is satisfied for the entity described by the ϵ model, then $\epsilon = 0$ or $\epsilon = 1$.*

Proof. See [49].

Theorem 5. *If Axiom 5 is satisfied for the entity described by the ϵ model, then $\epsilon = 0$ or $\epsilon = 1$.*

Proof. See [49].

For a detailed study of other aspects of the ϵ-model, we refer the reader to [48–53].

11.8 Reflections on the New Theory

In the previous section, we saw that the quantum probability structure can be explained as being a consequence of the presence of fluctuations on the interaction between the measurement apparatus and the physical entity under consideration. The amount and the nature of the distribution of the these fluctuations determines the amount of deviation of the probability structure from a classical Kolmogorovian structure. The quantum mechanical probability structure appears for a situation with the maximal amount of fluctuations with a uniform distribution. There exist intermediate situations 'in-between classical and quantum', where the amount of fluctuation is neither zero nor maximal, and we have shown that these intermediate situations cannot be described within an axiomatic approach where the 6 axioms are satisfied. Furthermore, it is Axioms 4 and 5 that are at the origin of this impossibility.

More specifically, the intermediate situations cannot be described when the nonclassical parts of the state property space are represented by Hilbert spaces as in standard quantum mechanics. This indicates that a satisfactory modelling of the classical limit will only be possible within a more general theory where Axioms 4 and 5 are relaxed. Since Axioms 4 and 5 are also the axioms that stand in the way of a description of the situations of a joint entity consisting of two separated quantum entities, we believe that these two axioms should really be relaxed and replaced by other axioms if we wish to proceed. Since Axiom 5 (the covering law) is equivalent to the existence of a vector space structure for the set of states of the physical entity under consideration, we believe that the superposition principle can no longer be seen as a general principle which is always satisfied.

11.8.1 Nonlocality is Nonspatiality

In this section, we would like to put forward another fundamental consequence of the hidden measurement hypothesis. Suppose we consider the hidden measurement explanation for the concrete situation of the state of a quantum entity described by a wave function $\psi(x, y, z)$, element of $L^2(\mathbb{R}^3)$, the Hilbert space of the square integrable complex functions of three real variables. Suppose that the wave function is well spread out and hence has the form of a Gaussian. In quantum jargon this means that the quantum entity under consideration is well delocalized. Suppose that we make a measurement of position, localizing the quantum entity within a region A of space. The new wave function, after the measurement, is given by

$$\phi(x, y, z) = \frac{\chi_A \circ \psi(x, y, z)}{\|\chi_A \circ \psi(x, y, z)\|} \,, \tag{11.52}$$

where χ is the characteristic function of the region A of space, and hence an orthogonal projection operator of the Hilbert space $L^2(\mathbb{R}^3)$. The quantum

entity after the measurement is much more localized, i.e., it is localized within the region A of space. From the hidden measurement hypothesis, it follows that the original wave function $\psi(x, y, z)$ describes a reality of the quantum entity under consideration, that is changed by the measurement into another reality of the quantum entity. Hence this is a reality that is not inside space. Even the wave function $\phi(x, y, z)$ describes a reality that is still not inside space. What happens is that, starting from a nonspatial state $\psi(x, y, z)$, the measurement transforms this state into another state $\phi(x, y, z)$ which is somewhat more spatial, because representing the quantum entity within the region A of space, but still nonspatial inside A. Only a delta function would describe a reality that is inside space, hence as a limiting case.

This means that the hidden measurement hypothesis has as a consequence that the 'locus' of a quantum entity is created by the position measurement itself and does not exist before the measurement has been performed. Nonlocality has to be interpreted as nonspatiality, and space cannot be seen as the theatre of all reality. Reality is much bigger than those parts of it that are contained inside space. Space should be interpreted as a structure which has emerged together with the macroscopic material entities that have emerged from the microscopic quantum entities, and it has emerged as 'their' space, meaning the 'space' in which these macroscopic entities exist and interact, as an emergent structure.

We have developed this picture in great detail in our group in Brussels and have called the philosophical view that corresponds to it the 'creation discovery view' [54–58].

11.8.2 Why General Relativity is not a Good Starting Point

The hidden measurement hypothesis also implies that the whole of the universe can still be supposed to be deterministic. The hidden measurement hypothesis does not imply this, but leaves it open as a possibility. In other words, quantum probabilities, which cannot be avoided if one describes a physical entity where there are intrinsic fluctuations present on the interaction between the measurement apparatus and the entity, can disappear if one focuses on a description of the whole of reality. In this sense, quantum structure and quantum probability appear as a consequence of considering a piece of the universe, and such a piece that it can only be studied by means of measurements that contain intrinsic fluctuations in their interactions with this piece. Classical entities, in this view, are special pieces of the universe, pieces such that there are measurements available that do not have these intrinsic fluctuations.

Having said this, it would mean that a deterministic theory like general relativity could eventually be the starting point for the new theory to be developed. This is true, except that there is a very important and somewhat hidden assumption in relativity theory, as it is currently set up, namely, the

assumption that the set of events has to be described by the points of a four-dimensional manifold, the points being interpreted as spacetime coordinates of the events. This assumption is the one that makes it impossible, in our opinion, to use the approach of general relativity theory as a starting point for the new theory to be developed. In [59–62], we have worked out an approach that could remedy this state of affairs. The main idea is that the set of happenings (we have introduced the concept 'happening' in [59–62] to replace the concept 'event', to make it clear that what we do is different from general relativity) does not a priori coincide with the set of spacetime points. In the next section we explain the ideas and approach presented in [59–62].

11.9 A Possible Framework

In this section we propose a framework that could be used to construct 'the new theory' that would have quantum mechanics and relativity theory as special cases. To elaborate this framework we make use of all the insights that an operational and realistic approach to quantum mechanics has given us. More specifically, we also use these insights for the case of relativity theory.

To do so we want to analyze the way in which we penetrate, clothe and decorate reality starting from our personal experiences. The main point we want to make is that there is a complex and largely forgotten process at the origin of how we penetrate, clothe and decorate reality, and it is by analyzing in detail this process that we will be able to see clearly through many of the paradoxical aspects of reality. Reality is out there. But the way that we know reality is through our experience of it. We order these experiences in a certain way, and are finally left over with a world view, in which what is 'real' has its specific place and function. We will see that physical theories, classical mechanics, quantum mechanics and relativity theory, have great difficulty in recovering and restating carefully what reality is, as we have introduced it within our pre-scientific personal world views. We will also see that a lot of the paradoxical aspects of our physical theories are due to a poor and fuzzy, and even sometimes wrong, understanding of this process.

11.9.1 Personal Experiences, Creations and Happenings

All the data that we gather about reality have come to us through our experiences. We consider an experience to be an interaction between a participator and a piece of the world. When the participator lives his or her experience, we say that this experience is present, and we call it the present experience of the participator. Note that we consciously use the word 'participator' instead of the word 'observer' to indicate that we consider the cognitive receiver to participate creatively in his or her cognitive act. For the situation of a measurement, we consider the experimentalist and his or her experimental

apparatus together to constitute the participator, and the physical entity under study is the piece of the world that interacts with the participator. The experiment is part of the experience.

Let us consider again the example that we mentioned in Sect. 11.4.2, but now in terms of experiences. Consider a piece of wood and two experiences that we can have with the piece of wood. One experience consists in testing whether the piece of wood 'burns well'. The test consists in applying a sufficient amount of fire to the piece of wood for a sufficient length of time to see whether it burns. Let us suppose that the piece of wood does indeed burn well, and let us call this experience E_1 : I apply fire to the piece of wood and it burns. Another experience that we consider tests whether the piece of wood floats on water. The test consists in putting the piece of wood on water and seeing whether it floats. Let us suppose that the piece of wood does indeed float on water, and let us call this experience E_2 : I put the piece of wood on water and it floats.

We have deliberately chosen these two experiences, because it is clear that we cannot experience them at once. If we tried to make the piece of wood burn and float on water, this would not work out well. So parts of both experiences are clearly incompatible, in the sense that they cannot be realized at the same time. Even though this is obvious for everybody that considers our example, there are parts of both experiences that we do consider to be present at the same time. Indeed, we do attribute two 'properties' to the piece of wood. One property is expressed by saying that the piece of wood has the property of burning well, and the other property is expressed by saying that the piece of wood has the property of floating on water. We believe that this single piece of wood, with which we do not have any of the two experiences E_1 and E_2, has at the same time the properties of 'burning well' and 'floating on water'.

Let us give a second example, that was originally introduced in [59]. Consider the following situation: I am inside my house in Brussels. It is night, the windows are shut. I sit in a chair, reading a novel. I have a basket filled with walnuts at my side, and from time to time I take one of them, crack it and eat it. New York exists and is busy. Let us enumerate the experiences that are relevant in this situation: E_3 (I read a novel), E_4 (I experience the inside of my house in Brussels), E_5 (I experience that it is night), E_6 (I take a walnut, crack it and eat it), E_7 (I experience that New York is busy).

As in the case with the piece of wood, where it is impossible to experience E_1 and E_2 at once, here also I do not experience all these experiences at once. On the contrary, in principle, I only experience one experience at once, namely my present experience. Let us suppose that my present experience is E_3 (I read a novel). Then a lot of other things happen while I am living this present experience. These things happen in my present reality. While I am reading the novel, some of the happenings that happen are the following: H_3 (the novel exists), H_4 (the inside of my house in Brussels exists), H_5 (it is

night), H_6 (the basket and the walnuts exist, and are at my side), H_7 (New York exists and is busy). All the happenings, and much more, happen while I live the present experience E_3 (I read a novel).

Why is the structure of reality such that what I have just said is evident for everybody? Certainly it is not because I also experience these other happenings. My only present experience is the experience of reading the novel. But, and this is the origin of the specific structure of reality as it appears in my world view, I could have chosen to live an experience including one of the other happenings in replacement of my present experience. Let me recapitulate the list of the experiences that I could have chosen to experience instead of my present experience: E_4 (I observe that I am inside my house in Brussels), E_5 (I see that it is night), E_6 (I take a walnut, crack it and eat it), E_7 (I take the plane to New York and see that it is busy). The same is true for the example of the piece of wood. While I live the experience E_1 (I apply fire to the wood and it burns), I could have lived, instead of this experience, the other experience E_2 (I put the wood on water and it floats), but I would have had to take another decision in my past, before I decided to start applying fire to the piece of wood.

These examples indicate how reality is structured within my world view. First of all we have to identify two main aspects of an experience: the aspect that is controlled and created by me, and the aspect that just happens to me and can only be known by me. Let us introduce this important distinction in a formal way. To see what I mean, let us consider the experience E_6 (I take a walnut, crack it and eat it). In this experience, there is an aspect that is an action by me, the taking and the cracking, and the eating. There is also an aspect that is an observation by me, the walnut and the basket. By studying how our senses work, I can indeed say that it is the light reflected on the walnut, and on the basket, that gives me the experience of walnut and the experience of basket. This is an explanation that can only be given now; it is, however, not what was known in earlier days when the first world views of humanity were constructed. But without knowing the explanation delivered now by a detailed analysis, we could see very easily that an experience always contains two aspects, a creation aspect, and an observation aspect, simply because our will can only control part of the experience. This is the creation aspect. For example, in E_3 (I read a novel) the reading is created by me, but the novel is not created by me. In general we can indicate for an experience the aspect that is created by me and the aspect that is not created by me. The aspect not created by me lends itself to my creation.

We can reformulate an experience in the following way: E_6 (I take a walnut, crack it and eat it) becomes E_6 (the walnut is taken by me, and lends itself to my cracking and eating) and E_3 (I read a novel) becomes E_3 (the novel lends itself to my reading). The taking, cracking, eating, and reading will be called creations or actions and will be denoted by C_6 (I take, crack

and eat) and C_3 (I read). The walnut and the novel will be called happenings and will be denoted by H_6 (the walnut) and H_3 (the novel).

> *A creation is that aspect of an experience created, controlled, and acted upon by me, and a happening is that aspect of an experience lending itself to my creation, control and action.*

An experience is determined by a description of the creation and a description of the happening. Creations are often expressed by verbs: to take, to crack, to eat, and to read, are the verbs that describe my creations in the examples. The walnut and the novel are happenings that have the additional property of being objects, which means happening with a great stability. Often happenings are expressed by a substantive.

> *Every one of my experiences E consists of one of my creations C and one of my happenings H, so we can write $E = (C, H)$.*

A beautiful image that can be used as a metaphor for our model of the world is the image of the skier. A skier skis downhill. At every instant he or she has to be in complete harmony with the form of the mountain underneath. The mountain is the happening. The actions of the skier are the creation. The skier's creation, fused in harmony with the skier's happening, is his or her experience.

11.9.2 How We Penetrate and Clothe Reality

Let us return to the collection of experiences: E_3 (I read a novel), E_4 (I observe that I am inside my house in Brussels), E_5 (I see that it is night), E_6 (I take a walnut, crack it and eat it) and E_7 (I take the plane to New York and see that it is busy). Let us now represent the way in which we penetrate and clothe the reality that is made out of this small collection of experiences. E_3 (I read a novel) is my present experience. In my past I could, however, at several moments have chosen to do something else and this choice would have led me to have a present experience other than E_3 (I read a novel). For example, one minute ago I could have decided to stop reading and observe that I am inside the house. Then E_4 (I observe that I am inside my house in Brussels) would have been my present experience. Two minutes ago I could have decided to stop reading and open the windows and see that it is night. Then E_5 (I see that it is night) would have been my present experience. Three minutes ago I could have decided to stop reading, take a walnut from the basket, crack it, and eat it. Then E_6 (I take a walnut, crack it and eat it) would have been my present experience. Ten hours ago I could have decided to take a plane and fly to New York and see how busy it was. Then E_7 (I go to New York and see that it is busy) would have been my present experience.

> *Even when they are not the happening aspect of my present experience, happenings 'happen' at present if they are the happening aspect*

of an experience that I could have lived in replacement of my present experience, if I had so decided in my past.

The fact that a certain experience E consisting of a creation C and a happening H is for me a possible present experience depends on two factors:

- I have to be able to perform the creation,
- the happening has to be available.

For example, the experience E_4 (I observe that I am inside my house in Brussels) is a possible experience for me, if:

- I can perform the creation that consists in observing the inside of my house in Brussels, in other words, if this creation is in my personal power,
- the happening 'the inside of my house in Brussels' has to be available to me, in other words, this happening has to be contained in my personal reality.

The collection of all creations that I can perform at the present I will call my present personal power. The collection of all happenings that are available to me at the present I will call my present personal reality.

I define as my present personal reality the collection of these happenings, the collection of happenings that are available to one of my creations if I had used my personal power in such a way that at the present I could fuse one of these creations with one of these happenings.

My present personal reality consists of all happenings that are available to me at present. My past reality consists of all happenings that were available to me in the past. My future reality consists of all happenings that will be available to me in the future. My present personal power consists of all creations that I can perform at present. My past personal power consists of all the creations that I could perform in the past. My future personal power consists of all creations I shall be able to perform in the future.

Happenings can happen 'together and at once', because to happen a happening does not have to be part of my present experience. It is sufficient that it is available, and things can be available simultaneously. Therefore, although my present experience is only one, my present personal reality consists of an enormous amount of happenings all happening simultaneously. This concept of reality is not clearly understood in present physical theories. Physical theories know how to treat past, present and future. But reality is a construction about the possible. It is a construction about the experiences I could have lived but probably will never live.

11.9.3 Material Time and Material Happenings

From ancient times, humanity has been fascinated by happenings going on in the sky, the motion of the sun, the changes of the moon, the motions of the planets and the stars. These happenings in the sky are periodic. By means of these periodic happenings humans started to coordinate the other experiences. They introduced the counting of the years, the months and the days. Later on watches were invented to be able to coordinate experiences of the same day. And in this sense material time was introduced into the reality of the human species. Again we would like to analyze the way in which this material time was introduced, to be able to use it operationally if later on we analyze the paradoxes of time and space. My present experience is seldom a material time experience. But in replacement of my present experience, I could always have consulted my watch, and in this way live a material time experience E_8 (I consult my watch and read the time). Hence, although my present experience is seldom a material time experience, my present reality always contains a material time happening, namely the happening H_8 (the time indicated by my watch), which is the happening to which the creation C_8 (I consult) is fused to form the experience E_8. It is in this way that time coordination is introduced into my personal reality.

> *The collection of all creations that I can perform at time t, I will call my personal power at time t. The collection of all happenings that are available to me at time t, I will call my personal reality at time t.*

Of course, as mentioned before, at time t, only one of my creations will be fused with one of the available happenings, which will lead to one experience that I live at time t.

11.9.4 Penetration in Depth and Width, Entity and Space

The two examples that we have considered give rise to seemingly different aspects of reality. We consider two happenings of the piece of wood, H_1 (the piece of wood entails the property of burning well) and H_2 (the piece of wood entails the property of floating on water), and two creations connected to this piece of wood, C_1 (I apply fire to the piece of wood) and C_2 (I put the piece of wood on water). Then the two experiences that we have considered are $E_1 = (C_1, H_1)$ and $E_2 = (C_2, H_2)$. The reason that we attribute the two properties 'the piece of wood burns well' and 'the piece of wood floats on water' to the piece of wood, is because we know that the two happenings H_1 and H_2 are available at once for one of the two creations C_1 or C_2 that I would choose to fuse to give rise to one of the experiences E_1 or E_2.

The example of the piece of wood shows us how we penetrate reality in depth, attributing properties to entities. Of course, we have to be aware that a more profound way of seeing this process of penetration in depth is the following. Certain happenings such as H_1 and H_2 cluster together, and

the entity 'piece of wood' is the collection of all these happenings. We call this way of clustering happenings together into an entity, our 'penetration in depth' of reality. All other properties of the piece of wood are linked to happenings that we have classified within the process of penetration in depth. For example the weight of the piece of wood, the fact that it is constituted of cells, of molecules, etc. The fact that the piece of wood constitutes an entity, is due to the fact that all these happenings do indeed cluster together. Of course, as we know, this clustering together is not absolute. We can break the piece of wood into two pieces of wood, and destroy some of the clustering.

Our other example is a typical example of what we will call 'penetration in width'. The happenings H_3, H_4, H_5, H_6 and H_7 are not clustered together, and indeed we will not consider them as part of an entity. At first sight we could say that these happenings are situated in 'space'. But again we have to correct ourselves. We should in fact proceed the other way around. Exactly as we have ordered the happenings that we collect by penetration in depth into an entity, because they are clustered together, we have ordered the happenings that we find by penetration in width into space. That is the way that we finally arrive at an image of my present reality consisting of space being filled up with different entities, where each entity is a cluster of happenings ordered by penetration in depth, and the different entities are spread over space, in this way attributing to the ordering of the happenings that we have collected by penetration in width.

It would be very fruitful to perform an analysis of reality where the division into penetration in depth and penetration in width were explicitly seen as two specific processes of penetration. In future work we hope to engage in such an analysis, because we believe that it will reveal deep and new insights into the nature of reality. In this paper we analyze some other aspects of this penetration. In our penetration in width, something remarkable occurs. We find entities, like our fellow human beings, occupying places in space other than the space we occupy ourselves. And we call these entities 'fellow human beings' because we believe that they also penetrate their personal reality in a similar way to the way that we penetrate our own personal reality. The big adventure of communication and dialogue starts here. Note that we do not experience something similar in our personal penetration in depth.

11.10 The Nature of the Present

Before being able to analyze the way in which we fuse personal realities into an encompassing intersubjective reality, we have to analyze the nature of space within our approach in a detailed way. To do so we have to take into account the results of relativity theory.

11.10.1 Relativity Theory and My Personal Present

Let us suppose that I am here and now in my house in Brussels, and it is 10 April 2003, 3 PM exactly. I want to find out 'what is the material reality for me now?'. Let us use the definition of reality given in the last section and consider a place in New York, for example at the entrance of the Empire State Building, and let us denote the center of this place by (x_1, x_2, x_3). I also choose now a certain time, for example 10 April 2003, 3 PM exactly, and let me denote this time by x_0. I denote the happening that corresponds to the spot (x_1, x_2, x_3) located at the entrance of the Empire State Building at time x_0 by H_9. I can now try to investigate whether this happening H_9 is part of my personal present.

The question I have to answer is: can I find a creation of localization l? In this case this creation is just the observation of the spot (x_1, x_2, x_3) at the entrance of the Empire State Building, at time x_0, that can be fused with this happening m. The answer to this question can only be investigated if we take into account the fact that I, who want to try to fuse a creation of localization to this happening, am bound to my body, which is also a material entity. I must specify the question introducing the material time coordinate that I ascertain by my watch. So suppose that I coordinate my body by the four numbers (y_0, y_1, y_2, y_3), where y_0 is my material time, and (y_1, y_2, y_3) is the center of mass of my body. We now apply our operational definition of reality. At this moment, 10 April 2003 at 3 PM exactly, my body is in my house in Brussels, which means that (y_0, y_1, y_2, y_3) is a point such that y_0 equals 10 April 2003, 3 PM, and (y_1, y_2, y_3) is a point, the center of mass of my body, somewhere in my house in Brussels. This shows that (x_0, x_1, x_2, x_3) is different from (y_0, y_1, y_2, y_3), in the sense that (x_1, x_2, x_3) is different from (y_1, y_2, y_3) while $x_0 = y_0$. The question is now whether (x_0, x_1, x_2, x_3) is a point of my personal present, hence whether it makes sense for me to claim that now, 10 April 2003, 3 PM, the entrance of the Empire State Building 'exists'. If our theoretical framework corresponds in some way to our pre-scientific construction of reality, the answer to the foregoing question should be affirmative. Indeed, we all believe that 'now' the entrance of the Empire State Building exists. Let us try to investigate this question in a rigorous way within our framework.

We have to check whether it was possible for me to decide somewhere in my past, hence before 10 April 2003, 3 PM, to change some of my plans of action, in such a way that I would decide to travel to New York, and arrive exactly on 10 April 2003, 3 PM at the entrance of the Empire State Building, and observe the spot (x_1, x_2, x_3). There are many ways to realize this experiment, and we will not go into details here, because we shall come back later to the tricky parts of the realization of this experiment. I could thus have experienced the spot (x_1, x_2, x_3) on 10 April 2003, 3 PM, if I had decided to travel to New York at some time in my past. Hence (x_0, x_1, x_2, x_3) is part of my reality. It is sound to claim that the entrance of the Empire State

Building exists right now. And we note that this does not mean that I have to be able to experience this spot at the entrance of the Empire State Building now, 10 April 2003, 3 PM, while I am inside my house of Brussels. I repeat again, reality is a construction about the possible happenings that I could have fused with my actual creation. And since I could have decided so in my past, I could have been at the entrance of the Empire State Building, now, 10 April 2003, 3 PM. Up to this point, one could think that our framework only confirms our intuitive notion of reality, but our next example shows that this is certainly not the case.

Let us consider the same problem as above, but for another point of time–space. We consider the point (z_0, z_1, z_2, z_3), where $(z_1, z_2, z_3) = (x_1, x_2, x_3)$, hence the spot we envisage is again the entrance of the Empire State Building, and z_0 is 11 April 2003, 3 PM exactly, hence the time that we consider is, tomorrow 3 PM. If I ask now first, before checking rigorously by means of our operational definition of reality, whether this point (z_0, z_1, z_2, z_3) is part of my personal present, the intuitive answer here would be 'no'. Indeed, tomorrow at the same time, 3 PM, is in the future and not in the present, and hence it is not real, and hence no part of my personal present (this is the intuitive reasoning). If we go now to the formal reasoning in our framework, then we can see that the answer to this question depends on relativity theory. Indeed, let us first analyze the question in a Newtonian conception of the world to make things clear.

Note that in a Newtonian conception of the world (which has been proved experimentally wrong, so here we are just considering it for the sake of clarity), my personal present coincides with 'the present', namely all the points of space that have the same time coordinate 10 April 2003, 3 PM. This means that the entrance of the Empire State Building tomorrow 'is not part of my personal present'. The answer is clear here and in this Newtonian conception, my present personal reality is just the collection of all (u_0, u_1, u_2, u_3) where $u_0 = y_0$ and (u_1, u_2, u_3) are arbitrary. The world is not Newtonian, this we now know experimentally; but if we put forward an ether theory interpretation of relativity theory (let us refer to such an interpretation as a Lorentz interpretation) the answer again remains the same. In a Lorentz interpretation, my present personal reality coincides with the present reality of the ether, namely all arbitrary points of the ether that are at time y_0, 10 April 2003, 3 PM, and again tomorrow the entrance of the Empire State Building is not part of my personal present.

For an Einsteinian interpretation of relativity theory the answer is different. To investigate this I must once again ask the question of whether it would have been possible for me to have made a decision in my past such that I would have been able to make (y_0, y_1, y_2, y_3) coincide with (z_0, z_1, z_2, z_3). The answer here is that this is very easy to do, because of the well known, and experimentally verified, effect of time dilatation. Indeed, it would for example be sufficient to go back a few weeks in my past, let us say to 15 March

2003, 3 PM, and then decide to step inside a spaceship that can move with almost the speed of light, so that the time when I am inside this spaceship slows down in such a way, that when I return with the spaceship to planet earth, still flying with a speed close to the velocity of light, I arrive in New York at the entrance of the Empire State Building with my personal material watch indicating 10 April 2003, 3 PM, while the watch that remained at the entrance of the Empire State Building indicates 11 April 2003, 3 PM. Hence in this way I make (y_0, y_1, y_2, y_3) coincide with (z_0, z_1, z_2, z_3), which proves that (z_0, z_1, z_2, z_3) is part of my personal present.

First one might say that in practice it is not yet possible to make such a flight with a spaceship. But this point is not crucial for our reasoning. It is sufficient that we can do it in principle. We have not yet made this explicit comment, but obviously if we have introduced in our framework an operational definition for reality, then we do not have to interpret such an operational definition in the sense that only operations that are currently possible, taking into account the present technical capabilities of humanity, can be performed. If we were to advocate such a narrow interpretation, then even in a Newtonian conception of the world, the star Sirius would not exist, because we cannot yet travel to it. What we mean by 'operational' is much wider. It must be possible, taking into account the actual physical knowledge of the world, to conceive of a creation that can be fused with the happening in question, and then this happening pertains to our personal reality.

11.10.2 Einstein versus Lorentz

We can come now to one of the points that we want to make, clarifying the time paradox that distinguishes an ether interpretation of relativity (Lorentz) from an Einsteinian interpretation. To remain clear about this question, we must return to the essential aspect of the construction of reality in our framework, namely, the difference between a creation and a happening. In order to clarify this, we first give another example.

Suppose that I am an artist and I consider once again my personal present, on 10 April 2003, 3 PM, as indicated on my personal material watch. I am in my house in Brussels and let us further specify: the room where I am is my studio, surrounded by paintings, of which some are finished, and others are still to be completed. Clearly all these paintings exist in my present reality, 10 April 2003, 3 PM. Some weeks ago, when I was still working on a painting that is now finished, I could certainly have decided to start to work on another painting, a completely different one, that does not now exist. Even if I could have decided this some weeks ago, everyone will agree that this other painting, that I never started to work on, does not exist now, 10 April 2003, 3 PM. The reason for this conclusion is that the making of a painting is a creation and not a happening. It is not the case that there is some 'hidden' space of possible paintings such that my choice several weeks ago to realize this other painting would have made me detect it. If this were to be the situation

with paintings, then indeed this painting would also exist now, in this hidden space. But with paintings this is not the case. Paintings that are not realized by the painter are potential paintings, but they do not exist.

With this example of the paintings we can explain very well the difference between Lorentz and Einstein. For an ether interpretation of relativity, the fact that my watch is slowing down while I decide to fly with the spaceship at nearly the speed of light and return to the entrance of the Empire State Building when my watch indicates 10 April 2003, 3 PM, while the watch that remained at the Empire State Building indicates 11 April 2003, 3 PM, is interpreted as a creation. It is seen as if there is a real physical effect of creation on the material functioning of my watch while I travel with the spaceship, and this effect of creation is generated by the movement of the spaceship through the ether. Hence the fact that I could observe the entrance of the Empire State Building tomorrow 11 April 2003, 3 PM, if I had decided some weeks ago to start travelling with the spaceship, only proves that the entrance of the Empire State Building tomorrow is a potentiality. Just like the fact that this painting that I never started to paint could have been here in my workshop in Brussels is a potentiality. This means that as a consequence the spot at the entrance of the Empire State Building tomorrow is not part of my present reality, just as the possible painting that I did not start to paint is not part of my present reality.

If, however, we put forward an Einsteinian interpretation of relativity, then the effect on my watch during the spaceship travel is interpreted in a completely different way. There is no physical effect on the material functioning of the watch – remember that most of the time dilatation takes place not during the accelerations that the spaceship undergoes during the trip, but during the long periods of flight with constant velocity nearly at the speed of light – but the flight at a velocity close to the speed of light 'moves' my spaceship in the time–space continuum in such a way that time coordinates and space coordinates get mixed. This means that the effect of the spaceship travel is an effect of a voyage through the time–space continuum, which brings me at my personal time of 10 April 2003, 3 PM to the entrance of the Empire State Building, where the time is 11 April 2003, 3 PM. And hence the entrance of the Empire State Building is a happening, an actuality and not just a potentiality, and it can be fused with my present creation.

This means that the happening (z_0, z_1, z_2, z_3) of 11 April 2003, 3 PM, entrance of the Empire State Building, is a happening that can be fused with my creation of observation of the spot around me on 10 April 2003, 3 PM. Hence it is part of my personal present. The entrance of the Empire State Building on 11 April 2003, 3 PM exists for me today, 10 April 2003, 3 PM. If we advocate an Einsteinian interpretation of relativity theory we have to conclude from the foregoing section that my personal reality is four-dimensional. This conclusion will perhaps not amaze those who have always considered the time–space continuum of relativity as representing the new

reality. However, now that we have defined very clearly what this means, we can start investigating the seemingly paradoxical conclusions that are often brought forward in relation with this insight.

11.10.3 The Process View and the Geometric View

The paradoxical situation that we can now try to resolve is the confrontation of the process view of reality with the geometric view. It is often claimed that an interpretation where reality is considered to be related to the four-dimensional time–space continuum contradicts another view of reality, namely the one where it is considered to be of a process-like nature. By means of our framework we can now understand exactly what these two views imply and see that there is no contradiction.

Let us now repeat what we mean in our framework when we conclude that my personal reality is four-dimensional. It means that, at a certain specific moment, which I call my present, the collection of places that exist, and that I could have observed if I had decided to do so in my past, has a four-dimensional structure, well-represented mathematically by the four-dimensional time–space continuum. This is indeed my personal present. This does not imply, however, that this reality is not constantly changing. Indeed it is constantly changing. New entities are created in it and other entities disappear, while others are very stable and remain in existence. This is in fact the case in all of the four dimensions of this reality.

Let us consider an example. We came to the conclusion that now, on 10 April 2003, 3 PM, the entrance of the Empire State Building exists for me while I am in my house in Brussels. But this is not a statement of deterministic certainty. Indeed, it is quite possible that by some chain of events, and without me knowing of these events, that the Empire State Building has disappeared, for example, because it has been rebuilt. Thus my statement about the existence of the entrance of the Empire State Building 'now', although almost certainly true, is not deterministically certain.[8] The reason is again the same, namely, that reality is a construction of what I would have been able to experience, if I had decided differently in my past. The knowledge that I have about this reality is complex and depends on the changes that go on continuously in it. What I know from experience is that there do exist material objects, and the Empire State Building is one of them, that are rather stable, which means that they remain in existence without changing too much. To these stable objects, material objects but also energy fields, I can attach the places from which I can observe them. The set of these places has the structure of a four-dimensional continuum. At the same time all these objects are continuously changing and moving in this four-dimensional

[8] We note that our example of the entrance of the Empire State Building was given in [60, 61], long before the terrible chain of events that led to the destruction of the Twin Towers came into being.

scenery. Most of the objects that I have used to shape my intuitive model of reality are the material objects that surround us here on the surface of the earth. They are all firmly fixed in the fourth dimension (the dimension indicated by the 0 index, and we should not call it the time dimension) while they move easily in the other three dimensions (those indicated by the 1, 2, and 3 indexes).

Other objects, such as electromagnetic fields, have a completely different manner of being and changing in this four-dimensional scenery. This means that in our framework there is no contradiction between the four-dimensionality of the set of places and the process-like nature of the world. When we come to the conclusion that the entrance of the Empire State Building, tomorrow, 11 April 2003, 3 PM also exists for me now, then our intuition reacts more strongly to this statement, because intuitively we think that this implies that the future exists, and hence is determined and hence no change is possible. This is an incorrect conclusion which comes from the fact that, over a long period of time, we have had the intuitive image of a Newtonian present, as being completely determined. We have to be aware of the fact that it is the present, even in the Newtonian sense, which is not determined at all. We can only say that the more stable entities in our present reality are more strongly determined as being there, while the places where they can be are always there, because these places are stable with certainty.

11.10.4 The Singularity of My Personal Present

We now come back to the construction of reality in our framework, which we have confronted here with the Einsteinian interpretation of relativity theory. Instead of wondering about the existence of the entrance of the Empire State Building tomorrow, 11 April 2003, 3 PM, I can also question the existence of my own house at the same place of the time–space continuum. Clearly I can make an analogous reasoning and then come to the conclusion that my own house, and the chair where I am sitting while reading the novel, and the novel itself, and the basket of walnuts beside me, etc., all exist in my present reality on 11 April 2003, 3 PM, hence tomorrow. If we put it like that, we are even more sharply confronted with a counterintuitive aspect of the Einsteinian interpretation of relativity theory. But in our framework, it is a correct statement. We have to add, however, that all these objects that are very close to me now, on 10 April 2003, 3 PM, do also exist in my present reality on 11 April 2003, 3 PM, but the place in reality where I can observe them is of course much further away for me. Indeed, to be able to get there, I have to fly away with a spaceship at nearly the velocity of light.

We now come to a very peculiar question that will confront us with the singularity of our reality construction. Where do I myself exist? Do I also exist tomorrow 11 April 2003, 3 PM? If the answer to this question is affirmative, we would be confronted with a very paradoxical situation. Because indeed I, and this counts for the reader too, cannot imagine myself to exist at different

instants of time. But our framework clarifies this question very easily. It is impossible for me to make some action in my past such that I would be able to observe myself tomorrow 11 April 2003, 3 PM. But if I had chosen to fly away and come back with the spaceship, it would be quite possible for me to observe now, on 10 April 2003, at 3 PM on my personal watch, the inside of my house tomorrow 11 April 2003, 3 PM. As we remarked previously, this proves that the inside of my house tomorrow is part of my personal reality today. But I will not find myself in it. Because to be able to observe my house tomorrow 11 April 2003, 3 PM, I have had to leave it. Hence, in this situation I will enter my house, being myself still at April 10, 2002, 3 PM, but with my house and all the things in it, being at April 11, 2002, 3 PM. This shows that there is no contradiction. In fact, if it comes to a much more common happening, the existence of the Empire State Building on 10 April 2003, 3 PM, nobody would even make this mistake, and think that, since to experience this existence I would have to take the plane and fly to New York 10 hours before, this would imply that I were in two places at once, in my house and in New York.

We can conclude this analysis of the nature of our personal present with the following observation. The nature of the present does not correspond very well to our intuitive idea of what this nature is. Indeed, intuitively we think of the present – our intuition being guided by a Newtonian world view – as the collection of all entities and their interactions that are inside space 'now', simultaneously with our personal 'now'. This is an incorrect conception. First of all the present has a four-dimensional structure and not a three-dimensional one, as would follow from a Newtonian world view. We note that this is mainly a consequence of the exactness of relativity theory. But secondly, and this is more important, and also not understood by scientists that are aware of relativity theory, the present is more like what we intuitively think about the future. It is not determinate in the simple way that we imagine. It is the collection of all happenings that I could have fused with one of my present creations, if I had decided to do something different in my personal past. This collection is determined is some way, but in a rather complicated way, which is equivalent to the way we intuitively feel the future to be determined in some complicated manner. Let us try to see the nature of this determination more clearly.

11.10.5 The Structure of My Personal Present

The basic structure of my personal present is represented in Fig. 11.4. I live the experience E_3 at time t_5, and this is my only present experience. But at times t_4, t_3, t_2, and t_1, I could have chosen another action than the one that leads to experience E_3, and that would then lead me to experience E_4, E_5, E_6, or E_7, respectively. That is the reason why happenings H_4, H_5, H_6, and H_7 exist at time t_5 in my personal present. We represent the situation analyzed in Sects. 11.10.1 and 11.10.2 in Fig. 11.5.

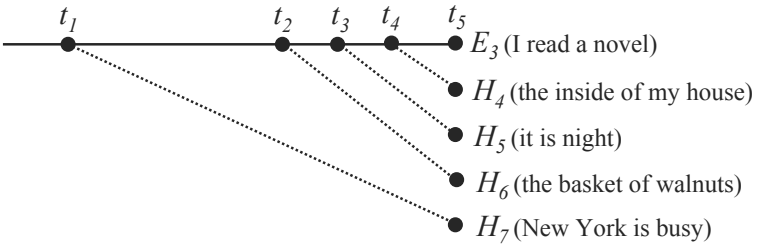

Fig. 11.4. A representation of my personal present. I experience the experience E_3 at time t_5. At time t_5 happenings H_4, H_5, H_6, and H_7 also happen, because I could have decided at times t_4, t_3, t_2, and t_1, respectively, to take another action than the one that leads me to experience E_3 at time t_5

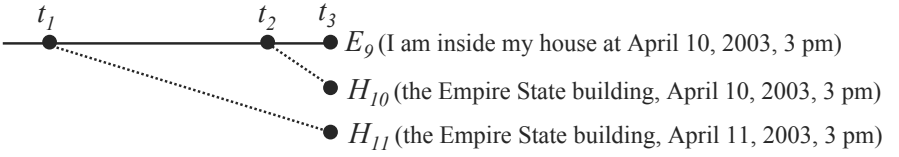

Fig. 11.5. A representation of my personal present. I experience the experience E_9 at time t_3. At time t_3 happenings H_{10} and H_{11} also happen, because I could have decided at times t_2 and t_1, respectively, to take another action than the one that leads me to experience E_9 at time t_3

We showed in Sect. 11.10.4 that the situation related to the fact that happenings that I would classify in the future in my intuitive view of the present are also in my present does not lead to a paradox of 'being able to meet myself'. In the next section, we will see that our intuitive view of the structure of the present has its roots in a further development of my personal present, namely the development connected with joining different personal presents into one intersubjective present.

11.10.6 Fusing of Different Personal Realities

In the foregoing we have analyzed the way I penetrate and clothe reality. Now an extra hypothesis comes into play. We know that the world is populated with other people, who also penetrate and clothe reality in this way, by forming their personal reality. Let us analyze some of the fundamental problems that appear when we attempt to fuse two such personal realities together into one encompassing reality.

A first remark we have to make is the following. If we consider Fig. 11.5, we see that the Empire State Building on 10 April 2003, 3 PM, as well as the Empire State Building on 11 April 2003, 3 PM, are both happenings that are part of my personal present on 11 April 2003, 3 PM. What is the problem with this. I myself am only present 'now', living my present experience. And a

moment later, this present experience has become one of my past experiences, and I am into a new present experience. My stream of experiences is hence a stream that moves from past to present, and what I have called my personal material time tracks this stream of experiences. If we now believe that the Empire State Building is also an entity, as I am, then we have to make the hypothesis that the Empire State Building exists only at one moment within its personal reality, namely its present. This means that there is a difference between 'exist' for the Empire State Building, within my personal present, and 'exist' for the Empire State Building within its own personal present. Note that we have already used the personal presents of the Empire State Building by indicating its times, 11 April 2003, 3 PM and 10 April 2003, 3 PM, which are, within the personal reality of the Empire State Building two happenings that are not in the same reality.

The problem becomes more obvious when another person is involved. So suppose that I consider my student Bart D'Hooghe, with whom I have discussed some of the problems treated in this article. Then in my personal present, Bart exists at all 'future' times that are indicated by his watch. In Bart's personal present, I exist at all future times that are indicated by my watch, as shown in Fig. 11.6.

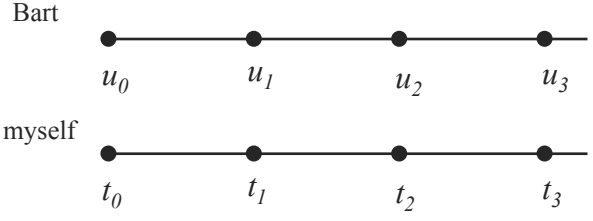

Fig. 11.6. Illustration of the personal present of myself and of Bart

We indicate the personal presents of myself, as they run through time, with a parameter t, which indicates the time on my personal watch, and in a similar way, we indicate the personal presents of Bart, as they run through time, with a parameter u, that indicates the time on Bart's personal watch. Then, Bart, at all times u_0, u_1, u_2, u_3 is part of my personal present at time t_0, while I myself, at all times t_0, t_1, t_2, t_3, am part of Bart's personal present at time u_0.

When time was Newtonian, the two types of existence could be fused together without difficulty. Indeed, in a Newtonian time frame, it will be the case that, for example, t_0 and u_0 can be said to be simultaneous, as well as t_1 and u_1, t_2 and u_2, and t_3 and u_3. And, if this is the case, we make a special slice within, for example, my personal present at time t_2, namely the slice that contains exactly Bart's personal present at time u_2 (and in a similar way t_0, t_1, and t_3, are identified with u_0, u_1, and u_3).

Due to relativity theory, such a simple synchronization of the two watches, my watch and Bart's, cannot be made. Even though we have the deep intuition that it should be possible, at my personal present at time t_1, to elect 'one' and 'only one' personal present moment of Bart (hence one value of u) to make it coincide with the t_1 of my personal present, we have to be aware that this deep intuition is not correct, given the above analysis about the way we penetrate reality. My personal present is the collection of all the happenings that I could have lived if I had decided something different in my past, something that would lead me to experience the specific happening that I am considering. The only way in which a subcollection of this collection of happenings could stand out and form a special subcollection, each of the happenings of the subcollection being in a certain sense more specifically related to my personal experience, is when it would be possible to classify the things that I could have done in my past in such a way that certain things stand out from the others. In the general scheme that we consider this is not possible.

We can put forward the image that the personal present of Bart, each moment of personal time of Bart following after another moment of personal time of Bart, runs through the set of happenings that are connected with Bart in my personal present. And in a similar way, I run through the collection of happenings that are connected with me in the personal present of Bart. It is still an unsolved problem, linked to the problem of synchronization in relativity theory [63,64], whether this view can be upheld. We believe that this problem should be analyzed, taking into account the two subtle and different notions of reality that we consider in this paper. We intend to investigate this problem in future work.

11.11 Conclusion

We believe that the new theory that will have quantum mechanics and general relativity as special cases and also explain why they are special cases will have to be formulated within a structural and mathematical context that is very different from those of existing physical theories. The operational axiomatic approach that we have proposed in Sects. 11.2, 11.3, 11.4, 11.5 and 11.6 can deliver a framework for this new theory. This approach makes it possible to formalize the subject–object actions that are possible in our world in the most general way. In this sense it is similar to general relativity in that it formalizes the 'geometrical beings' in our world in the most general way, but it is more general, because geometrical beings are special cases of subject–object interactions, where the subject–object action is reduced to a simple observation.

Since we have identified two failing axioms within the set of six axioms that make the general approach equivalent to standard quantum mechanics with superselection rules, in relation with well-defined physical situations, i.e.,

the situation of separated quantum entities (see Sect. 11.6.1), and the situation of the continuous transition from quantum to classical (see Sect. 11.7.3), it will not be possible to use standard quantum mechanics for this general enterprise, its mathematical structure being too limited. This is why the operational axiomatic approach may be an essential tool to build the necessary structure for the new theory.

The explanation we have developed for quantum probability, presented in Sect. 11.7, shows that the non-Kolmogorovian nature of quantum probability can be explained as being the consequence of a lack of knowledge about the interaction between the measuring apparatus and the physical entity under consideration, hence due to the presence of fluctuations in this interaction. This means that there is no incompatibility with the hypothesis of a universe as a whole that is deterministic, and the presence of quantum probability, as an irreducible probability for the description of a piece of this universe that we call physical entity.

The third important aspect of our approach is the interpretation of nonlocality. From our explanation for the quantum probability structure, it follows that nonlocality has to be interpreted as nonspatiality. Nonlocal states of a quantum entity are nonspatial states, meaning literally that a quantum entity in such a state is not inside space. As a consequence, space cannot be seen as an all-embracing theatre for reality, but must be interpreted as a macroscopic structure that has emerged in the same process of emergence of macroscopic physical entities from the micro-world. Space is 'the space' of the macroscopic physical entities and not of the microscopic quantum entities. In our opinion this is the main reason why the global and fundamental approach of general relativity cannot incorporate the quantum world. One of the steps of general relativity is indeed to identify the set of all events with the set of all time–space points of the four-dimensional time–space continuum (Sect. 11.8).

In Sects. 11.9 and 11.10 we make the first steps towards a theory that uses the insights and methodology of the operational axiomatic approach to quantum mechanics within a scheme which aims at a description in the style of relativity theory.

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12 What Is Probability?

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What is probability? Physicists, mathematicians, and philosophers have been engaged with this question since well before the rise of modern physics. But in quantum mechanics, where probabilities are associated only with measurements, the question strikes to the heart of other foundational problems: what distinguishes measurements from other physical processes? Or in more formal terms: when are the unitary dynamical equations suspended in favour of probabilistic ones?

This is the *problem of measurement* in quantum mechanics. The most clear-cut solutions to it change the theory: they either add hidden variables (as in the pilot-wave theory), or they give up the unitary formalism altogether (as in state-reduction theories). The two strategies are tied to different conceptions of probability: probability as in classical statistical mechanics (as formulated by Boltzmann, Gibbs and Einstein), and probability as in Brownian motion (with the dynamics given by a stochastic process, as formulated by Einstein and Smulochowski). The former is sometimes called *epistemic* probability, as classical mechanics is deterministic: probabilities arise as a consequence of incomplete knowledge, or of incomplete description. It is the latter, stochastic, probability that is usually thought the more fundamental, as it enters directly into the equations.

Indeterminism encouraged a late-comer to the philosophy of probability, Popper's *propensity* approach [13], in which probabilities are identified with certain kinds of properties – 'dispositions' – of chance setups. But whether thought of in terms of incomplete descriptions, or as propensities, these kinds of probability have puzzled philosophers. By way of contrast, probability as degree of subjective expectation, or subjective weighting, is much clearer: if that is what probability is one can explain why it obeys the rules that it does (I shall come back to this point later). Neither the epistemic probabilities of classical statistical mechanics, nor the objective probabilities of a stochastic dynamics as in Brownian motion, are happily thought of as subjective expectations. They are grounded, surely, in facts about physical states of affairs, independent of persons altogether. It is true that epistemic probability is often thought of in terms of ignorance, or incomplete knowledge, but it is precisely this link with purely subjective considerations that is so puzzling from the point of view of its role in thermodynamics. Heat transfers, surely,

take place according to thermodynamical laws, independent of whether anyone is looking, and of what anyone knows. The point is all the more evident in the case of stochastic probability. Epistemic as well as stochastic probabilities are objective rather than subjective quantities. What, then, are the physical facts that make true some probability statements but not others? What are we trying to get right about when we make judgements of objective probabilities? It is not informative to say it is that there are *chances* that are thus and so; the difficulty, that has long bedevilled the philosophy of probability, is to say what chances could possibly be.

There are well-known, failed, candidates for the role. The most commonplace looks to the evidence for probability statements, the observed statistical data. Of course, if the question is what chances are, rather than what we believe them to be, we should consider unobserved data as well; in the most simple-minded approach, these two are simply identified – specifically, chances are identified with long-run relative frequencies. But the objections to doing this are obvious. How long is long enough? In the short run we cannot expect chance and relative frequencies to line up exactly, whilst the infinite limit is never actually reached; so what, precisely, are the chances? Presumably a function of the number of trials and the actual relative frequencies; but what function?

One can prove that for repeated trials the relative frequencies of outcomes will approximately match the probability of each outcome (assumed independent), but only in the sense that if $p(n, \delta)$ is the probability that they differ by more than $\delta > 0$, for n trials, then $\lim_{n\to\infty} p(n, \delta) = 0$ (the law of large numbers). But this is not to identify probabilities with any actual relative frequencies; so long as n is finite, we still have to do with probabilities [the quantities $p(n, \delta)$]. What this and the various other laws of this kind show, rather, is that the concept of probability enters into the very relation between statements of evidence and statements about chance; statistical evidence bears out a probability claim only to some degree – where 'some degree' is some degree of probability.

Could the latter probability be subjective rather than objective probability? It could – but only if we assume that subjective probability, at least in such cases, agrees with chance: for we calculate the probability for matching statistics with chances by using those same chances. What underwrites this assumption? This is a question that would arise even if one knew what chances really were; why should subjective expectations be set equal to objective probabilities? Put to one side worries about long-run frequencies not being long enough; why, given that the long-run frequency of some outcome is thus and so, should we subjectively expect that outcome to that degree on the very next trial?

We have two questions:

(1) What, physically, is objective probability (chance)?
(2) Why should subjective probability track chance?

Because of its importance in the philosophy of probability and of rational action, the principle that subjective probability should track chance – or the slightly more involved principle that it should, however much additional information one may have, short of knowledge of what the actual outcome will be, has been dubbed the *principal principle* [11]. We are sure this principle is true, but we are at a loss to say why. Indeed, failing an account of what chances are, it is hard to see just how the principle *could* be justified; for it ought to be facts about physical states of affairs that dictate our subjective expectations of future contingencies. What are those facts? The two questions are interdependent.

One can rest on the authority of science. One can say that it is a requirement of any theory of rationality that our beliefs should be based on our best scientific theories. One can take it as an extension of this that we should tailor our subjective expectations of chance outcomes to whatever our best theory says are their chances; the principal principle would then be part of any theory of rational action worthy of the name. Maybe so, but then it will be an unexplained part. It is also a very large part of the theory of practical reasoning, perhaps the largest part (the part that deals with physical contingencies). One would, in the light of our best theories in physics, like to do better. The challenge is to say what it is about the world that makes statements of objective probability true, and why, given such states of affairs, we should act accordingly, with subjective probabilities fixed by the objective ones.

Questions (1) and (2) are now the most important ones in the philosophy of probability. It was not always so; Popper, when he proposed to abandon the link between long-run frequency and chance, wanted an account of probability that made sense of the single-case and that made no reference to human knowledge. Worthy aims; but to suppose that the chance of an outcome of an experiment is a 'disposition' of that experiment (and more generally, that the chance of an event given a certain chance setup is a 'disposition' of that setup) in itself solves very little; for what, physically, are these 'dispositions'? Popper was never able to say in terms of categorical physical properties – properties that are not themselves 'chancy' or equally in need of explanation. Nor was he able to in classical physics. Even in classical games of chance, where chances are directly related to the symmetries of a chance setup – the six faces of a die, the two sides of a coin – dynamics comes into it. Throw the die or the coin just so, and the statistics, the evidence for the chances, can be anything you like. The dynamics, it seems, can override chance.

This worry would seem to arise in any deterministic theory. It is made marginally more palatable by putting it in terms of initial conditions rather than dynamics (that certain initial conditions are 'typical'). Better still, adopt the picture of a probability distribution as a measure on an ensemble (usually infinite) of hypothetical physical systems, a picture in which dynamics is absent altogether. One can understand its appeal, but only from this narrow

perspective. What it is about the actual world that makes the probabilities what they are is never explained. Why we are supposed to have expectations about the real world because of the values of a measure on a fictitious ensemble of imaginary worlds is never explained.

(1) is evidently a hard question, but it is at least a physical question; it is not so clear that (2) is. It would be understandable were physicists to limit themselves to (1), to be answered by physics as usual (as in saying what temperature really is, or what solidity really is); (2) can be left to the philosophers. But we are interested in probability, above all, in quantum mechanics, and there one is up against the problem of measurement. When it comes to the problem of measurement physics is not its usual self. As we shall see, it turns out that the best answer to (1) so far in evidence also provides an answer to (2) – but at a price.

Our story must proceed in stages. We begin with orthodoxy; next we consider alternatives to quantum mechanics. From then on we consider probability from the point of view of decoherence theory and the unadulterated formalism.

12.1 Orthodoxy

12.1.1 Gleason's Theorem

Consider a system a with Hilbert space H^a and inner product $\langle \cdot, \cdot \rangle$. For $\phi \in H^a$ and any projector $\widehat{P}$ on H^a, the probability $\mu(\phi, \widehat{P})$ is defined by the rule (the *Born rule*):

$$\widehat{P} \text{ is measured in } \phi \quad \Longrightarrow \quad \mu(\phi, \widehat{P}) = \frac{\langle \phi, \widehat{P}\phi \rangle}{\langle \phi, \phi \rangle} \, .$$

From the right-hand side we see that μ is additive and indeed countably additive over any partitioning $\{\widehat{P}_k\}$ of H^a (any pairwise orthogonal set of projectors summing to the identity). Looking to the left-hand side, we suppose that each partitioning corresponds to a particular kind of experiment (one that *measures* $\widehat{P}$). This is where the 'intent' of the experiment or the notion of the 'observation' that is made comes in; it has proved to be very hard to make do with a purely physical specification of the apparatus.

We need a few more definitions in order to state Gleason's theorem. Given a density matrix ρ (positive, self-adjoint, and with unit trace) in place of a vector in H^a, the rule is:

$$\widehat{P} \text{ is measured in } \rho \quad \Longrightarrow \quad \mu(\rho, \widehat{P}) = \mathrm{Tr}(\rho \widehat{P}) \, ,$$

where Tr is the trace, yielding a weighted sum over probabilities as defined for the pure case. The set of all projectors on H has an algebraic structure defined by subspace inclusion (a partial ordering). Using it one can define

the meet ('and') and join ('or') operations, and under these it is a lattice. It is not of course a Boolean lattice (for which the meet and join operations are distributive), but it has (infinitely) many Boolean sublattices, and it is natural to ask that a probability function on this lattice should be additive on each Boolean sublattice of the total lattice. This is the condition of Gleason's celebrated theorem [8]:

Theorem 1 (Gleason). *Let f be any function on projectors on a Hilbert space H of dimension $d > 2$ to the unit interval which is additive for any set of pairwise disjoint projectors on H. Then there exists a unique density matrix ρ such that for any $\widehat{P}$ on H, $f(\widehat{P}) = \mathrm{Tr}(\rho\widehat{P})$.*

Gleason's theorem is a derivation of part of the Born rule, but of course it says nothing about 'measurements' or 'experiments'; nor, on reflection, is the premise of the theorem so clearly motivated. It is assumed that the probability for an outcome $\widehat{P}$ belonging to one sublattice is the same when $\widehat{P}$ is considered as a member of another. The assumption appears innocuous, but it has non-trivial consequences. For example, let $\widehat{P}_i$ project onto the subspace spanned by $\chi_i \in H^a$, $i = 1, 2$, and let $\widehat{P}_\pm$ project onto $\chi_1 \pm \chi_2$; then $\{\widehat{P}_1, \widehat{P}_2\}$ generates one Boolean sublattice and $\{\widehat{P}_+, \widehat{P}_-\}$ another. Yet if $f(\widehat{P}_1) = f(\widehat{P}_2) = 0$, then by additivity $f(\widehat{P}_1 + \widehat{P}_2) = f(\widehat{P}_+ + \widehat{P}_-) = 0$; and since f is positive, from additivity again it follows that $f(\widehat{P}_+) = f(\widehat{P}_-) = 0$. So probabilities for one family of projectors constrain those for another, even though the two do not commute, i.e., $[\widehat{P}_i, \widehat{P}_\pm] \neq 0$. Should constraints like this be imposed that relate measurements on non-commuting operators? Like that other celebrated theorem in the foundations of quantum mechanics (von Neumann's no-go theorem for hidden variables), the condition of Gleason's theorem may be physically unmotivated.

Of course the premise can be taken as the expression of a phenomenological principle. It is true as goes the statistics of actual experiments; the statistics of an outcome are the same whatever other quantity (so long as it is represented by a commuting operator) is measured. Gleason's theorem, we may suppose, shows us how a phenomenological principle implies a certain mathematical representation of probabilities in quantum mechanics, rather as Kelvin showed how thermodynamical laws imply a certain representation of temperature; and rather as (although it is a bit of a stretch) Einstein showed how the relativity principle and the light speed principle imply a certain representation of geometry.

The comparison with thermodynamics is not an altogether happy one. It reminds us that statistical data may well be consistent with deterministic theories. This point was important to Bell [1], who was sympathetic to the idea of introducing hidden variables in quantum mechanics. He sought to understand the observed statistics in terms of averages over states ruled out by Gleason's theorem, dispersion-free states, in which every projector has value zero or one. And Bell noticed that they are excluded by a much simpler

argument than Gleason's: from the result just proved, and from the fact that (again from additivity) if $f(\widehat{P}_\chi) = 1$ and if ϕ is orthogonal to χ then $f(\widehat{P}_\phi) = 0$, it follows immediately that if $f(\widehat{P}_\chi) = 1$ and $f(\widehat{P}_\phi) = 0$ then χ and ϕ cannot be too close ($|\chi - \phi| > |\phi|/2$). So there can be no dispersion-free states, for if dispersion-free $f(\widehat{P}_\chi)$ must change from 1 to 0 as χ is continuously rotated into ϕ, so it must change for vectors that are arbitrarily close.

According to Bell what is wrong with Gleason's additivity assumption, at the level of the single case, is that it ignores a clear possibility that cannot in principle be ruled out on experimental grounds. In the single case, the values assigned to $\widehat{P}_1$ and $\widehat{P}_2$ may be zero (and that can be discovered experimentally), but one cannot simultaneously measure $\widehat{P}_+$ or $\widehat{P}_-$ so one can draw no conclusion as to the value assigned to them in that context.

12.1.2 Bohr's Copenhagen Interpretation

Bell's reasoning was faithful to Bohr's *principle of complementarity*, according to which quantum mechanical phenomena cannot be defined independently of an experimental context. Given this and the fact that experiments that measure non-commuting quantities cannot simultaneously be performed, the way is open for results of experiments to defy classical reasoning altogether; it is possible (this is Bohr's principle of complementarity) that results obtained under incompatible experiments cannot be consistently fitted together according to the classical ideal of explanation. It is this that makes room for genuine novelty, in quantum experiments, according to Bohr. This argument was made repeatedly in Bohr's published writings.

It is therefore embarrassing, to Bohr if not to Bell – because Bohr was out to interpret quantum mechanics rather than to change it – that by this reasoning a loophole is opened up in Gleason's theorem. It may be that every dynamical quantity has a well-defined value (and every projector has the value 0 or 1) but that values for commuting quantities not in fact measured differ from the values that they would have had if they were. Such a theory is called a *contextual* hidden variable theory.

The same applies to the Kochen–Specker theorem (in effect a strengthening of Bell's theorem). Bohr's complementarity opens a way out for dispersion-free states in that case as well – and a way to understand the quantum probabilities as describing only the statistics.

Leaving so much open, the orthodoxy as we have it from Bohr offers no definite account of probability and no justification for the Born rule. And even if it could be used to underwrite Gleason's premise, it would give no answer to (1). For according to Bohr, the state is not something physically real; the squared modulus of the amplitude is not a categorical physical quantity. For all that it suggests a link between (1) and (2), it does so at the expense of (1). The Born rule gives the probabilities for the commuting family of $\widehat{P}$s that the experiment is 'intended' to measure. In answering (1), one wants to

dispense with the *intentions* (for the probabilities we are after are supposed to be objective rather than subjective). What is it from a purely physical point of view about an experimental apparatus that dictates that it is one set of probabilities that is relevant to the outcomes rather than another? What is the correct choice of sublattice, or equivalently, of basis?

Insisting as he did that the apparatus must be described in classical terms, Bohr was not entirely at a loss to answer this question. Example by example, he tried to show that the basis was dictated by some concrete feature of the apparatus – for example, by whether or not some shutter, screen or diaphragm was bolted to the laboratory bench. This was supposed to work in tandem with Bohr's further thesis that in quantum mechanics one never really went beyond classical concepts (or one or another of a complementary set of classical concepts); that there were, in effect, *no* genuinely quantum mechanical concepts, or none that could function in explanations in the way that classical concepts did. But here Bohr was obviously at a disadvantage; it was part and parcel of complementarity that one could not recover the classical ideal of explanation when it came to atomic phenomena; it was an *a priori* prejudice on Bohr's part that genuinely quantum concepts could never be found that would do better than the fragments of classical physics that Bohr did vouchsafe to us.

Few were prepared to follow Bohr with his analysis of quantum mechanics, and eventually of quantum field theory, in terms of fragments of classical physics. Even restricted to the analysis of measurements, his systems of trapdoors, levers and springs seemed baroque; he never was able to establish any hard-and-fast connection between what was bolted to what and the observable that was supposed to be measured. The doctrine of incomplete explanation was unsuccessful in the other areas where Bohr hoped it would deliver, in biology and psychology. It never offered any insight into the nature of probability.

12.2 Alternatives to Quantum Mechanics

12.2.1 Pilot-Wave Theory

The alternatives to quantum mechanics are well known. To begin with pilot-wave theory, which retains the unitary equation for the state and supposes indeed that quantum mechanics is universal (so there is a wave function for the universe), one has dispersion-free states, as Bell wanted, but only for certain dynamical variables (namely configuration space variables, the relative positions and relative velocities). Correspondingly, one has an additional equation – the guidance equation – that dictates the allowable trajectories through each point of configuration space. Consistent with the Bell–Kochen–Specker theorem, definite values are not attributed to every self-adjoint operator in quantum mechanics, independently of context. In fact most operators

are not assigned values at all (for example, only in certain contexts is any component of spin assigned a value). And where a component of spin is assigned a value (in the context of, say, a Stern–Gerlach experiment, for a particular orientation of the magnetic field), no value for any other component of spin is defined.

This is to rehabilitate Bohr's reasoning about experiments (although it can hardly be said to lend support to his general philosophy). But the more clear-cut interpretation of the pilot-wave theory is to suppose that the only real physical quantities are configuration space variables (relative positions and velocities); that all the rest are artifacts of experiments. Dispersion-free states for these quantities are not contextual, except in the sense that they may of course (by the non-locality of the theory) change when the macroscopic apparatus is changed. Bell himself seemed to advocate this position [2].

So how does probability get into the picture? Much as it does in classical statistical mechanics: one probability distribution on configuration space is favoured (as given by the Born rule) for much the same reason that Liouville measure on phase space is favoured in classical Hamiltonian mechanics. The Born rule is said to be the 'equilibrium' distribution ('quantum equilibrium'). Once in equilibrium, systems cannot be reliably prepared in the dispersion-free states allowed by the theory (it is this that hides the non-locality). This smacks of conspiracy; but given equilibrium, the situation is in one respect better than in classical statistical mechanics. At least we are in a position to answer (1): chances are determined by certain categorical properties in the world (the squared norms of the components of the wave function with respect to the position basis) – assuming, as proponents of the pilot-wave theory usually do, that the wave function is physically real.

Now note the two disadvantages of this approach. The first is that if this is the answer to question (1), question (2) seems entirely impenetrable. It is hard to see why our subjective expectations should be concerned with these amplitudes squared. The trajectory in phase space can after all be chosen to give you any statistics you like (leaving the amplitudes completely unchanged); if it is the trajectory we are concerned with – this is what picks out all the actual things that happen as opposed to those that do not – why should something completely independent of the trajectory, the amplitudes, be of any relevance?

The second turns this objection around. Why, in any case, suppose the physical probabilities are given by the Born rule? Maybe they float free of it, as classical probabilities can float free from equilibrium distributions in statistical mechanics [18]. There, non-equilibrium is not ruled out by fiat as somehow illegitimate (classically the universe is far from equilibrium). But if so, and there is no intrinsic connection between physical probabilities and the amplitudes, what do probabilities correspond to? And we are back to square one.

12.2.2 State-Reduction Theory

In pilot-wave theory state reduction is 'effective', as components of the state ('empty waves') irrelevant to the guidance equation can be discarded. The alternative is to build it into the dynamics directly. Here, unlike the case of pilot-wave theory, a considerable body of new work in physics and mathematics was needed. The mathematical foundations were laid by Smulochowski, Wiener and von Neumann in the 1920s. Investigations in quantum optics in the late 1950s made use of probability measures in quantum mechanics generalized to sequential sample spaces. They also generalized to continuous sampling; in the mid-1960s Nelson showed how the Schrödinger equation may be related to a continuous Markovian stochastic process. By the end of the 1970s, with investigations in open quantum systems theory by Davis and his collaborators, it was clear how to construct stochastic models to mimic the behaviour of subsystems subject to arbitrary quantum evolutions. The first stochastic theory demonstrably equivalent to standard quantum mechanics across a wide range of applications (all of them, however, non-relativistic) was proposed in 1986 by Ghirardi, Rimini, and Weber. These theories are therefore of comparatively recent origins; what do they say about probability?

The Schrödinger equation is replaced by a stochastic vector-valued dynamical process. In what is perhaps the most elegant example, the continuous state-reduction model [9], this process is assumed to be Markovian and to take the form:

$$\mathrm{d}\psi = \left[\widehat{Q}(\psi)\mathrm{d}t + \widehat{\boldsymbol{R}}(\psi)\cdot\mathrm{d}\boldsymbol{B}\right]\psi\ . \tag{12.1}$$

Here $\widehat{Q}$ and $\widehat{\boldsymbol{R}}$ are self-adjoint operators on H that depend on the state ψ, $\boldsymbol{B}$ is a smooth Markov process with components B_k, $k = 1, 2, 3$, satisfying (here γ is a new fundamental constant):

$$\overline{\mathrm{d}B_i(t)} = 0\ , \quad \overline{\mathrm{d}B_k(t)\mathrm{d}B_j(t)} = \delta_{kj}\gamma\mathrm{d}t\ .$$

The over-bar denotes averaging with respect to the underlying probability space of the Markov process. Each B_k is a map from its index set (time) to random variables (measurable functions) on this space. A probability distribution here as always – mathematically – is given by a measure on a σ-algebra of sets (a Borel space). What this measure corresponds to physically [question (1)], and why these values of the measure should dictate subjective probabilities [question (2)], is as obscure as ever.

Is there then no improvement in clarity about probability in this theory? It would be odd if so; with a stochastic, indeterministic theory, the probabilities are supposed to be in the theory at the ground level. One would think that there would be a clearer ontological basis to them than in a deterministic theory. There is this difference: whereas in classical statistical mechanics the measure is defined over a space of one-time data (Cauchy data), each point of

which encodes an entire trajectory, in a stochastic theory the measure is defined over the space of trajectories (histories). This, mathematically speaking, is the whole of the difference between a deterministic and an indeterministic theory. The customary distinction between epistemic and non-epistemic interpretations of probability is more rooted in temporal matters than first appears. One may say of a stochastic theory as much as of a deterministic one that probabilities are epistemic: they reflect one's ignorance of which history is ours. The difference is that in the deterministic case one must be ignorant of much more, if probabilities are to make any sense: one cannot know even a single instance of our actual history.

What of the Born rule? It is built into the dynamics, (12.1), in the dependence of the operators $\widehat{Q}$ and $\widehat{\mathbf{R}}$ on the state. This is of the form:

$$\widehat{\boldsymbol{R}}(\psi) = \widehat{\boldsymbol{A}} - \langle \psi, \widehat{\boldsymbol{A}}\psi \rangle \,, \quad \widehat{Q}(\psi) = -\frac{\mathrm{i}}{\hbar}\widehat{H} - \widehat{\boldsymbol{R}}\cdot\widehat{\boldsymbol{R}} \,,$$

where $\widehat{H}$ is the Hamiltonian, and $\widehat{\boldsymbol{A}}$ is a commuting triple of self-adjoint operators transforming as a Galilean 3-vector (remember we only have a non-relativistic stochastic theory). Different choices of the dependence of $\widehat{Q}$ and $\widehat{\boldsymbol{R}}$ on ψ will lead to a stochastic process yielding statistics at variance with the Born rule. It is therefore clearer in a state-reduction theory than in pilot-wave theory why the squared amplitudes matter to probabilities, since they enter directly into the dynamics. There is no longer the possibility that the dynamics can reliably drive an initial state through a sequence of states in which the relative frequencies of outcomes is independent of the squared amplitudes (the Born rule). But everything now hangs on the notion of a 'reliable' dynamics; the point is the same as before, that one might as well speak of a 'typical' trajectory (and of probability as a matter of ignorance as to which trajectory is one's own). An d, indeed, from a purely mathematical point of view, one is back to a definition of probability as a measure on history space. What it is about a particular history (the one that is actually realized) that makes it true that the probability of a chance outcome has a particular value is never explained. The link with subjective probabilities is as opaque as ever.

12.3 Decoherence Theory

Returning to the conventional formalism, there are two important lessons that we can draw from pilot-wave theory and state-reduction theory. In each there is a universal state that applies to closed systems; and in each there is state-reduction – merely 'effective' in the former theory, fundamental in the latter – yielding states which are well-localized in configuration space (there is a *single* preferred basis. Even in Bohr's philosophy localization, or 'space-time coordinatization' as he called it, played a fundamental role in any

measurement process). With this it is enough to recover all extant experimental data (at least in the case of non-relativistic applications of quantum mechanics). These are the lessons that are taken to heart in decoherence theory.

Decoherence theory was powered by investigations in open quantum systems theory just as the theory of quantum stochastic processes was. It has in common with pilot-wave theory the assumption that the unitary equations of motion are both fundamental and universal, and it has in common with state-reduction theories many of the same equations, but derived as *effective* equations, concerning only certain dynamical variables and degrees of freedom of a system. In any given case of decoherence, probabilities are defined only for a certain basis or Boolean sublattice of projectors. For systems at ordinary temperatures in which massive degrees of freedom are weakly coupled to large numbers of much lighter ones, the basis for which simple, effective equations exist for the massive ones is always (approximately) the same: it is given by projectors onto well-localized regions of the configuration space of the massive degrees of freedom, at least down to atomic dimensions and time scales of the order of classical thermal relaxation times. Thereby, so long as velocities and momenta are obtained by averaging over these time scales, one obtains coarse-grained trajectories in configuration space and phase space as well. These are classically perspicuous descriptions of atomic and molecular processes (with quantum correction terms added).

Decoherence plays a role in all the major schools of foundations in quantum mechanics. It matters to the pilot-wave theory, for it explains just where and when one can use the reduced state, discarding components of the state (the 'empty waves') for the purposes of actually applying the theory and computing trajectories. It matters to state-reduction theory, which stochastically degrades every component of the state save one. Degrade them too soon (before they have decohered) and the predicted probabilities will differ from those of quantum mechanics. And it matters to the *consistent* histories approach. Histories, in this formalism, are ordered sequences of projectors $\widehat{P}_k$ (of some given resolution of the identity), interpreted as a sequence of events in time, as specified by some discrete time variable and the resolution of the identity used at each time. They are consistent insofar as the probabilities for each history are non-interfering, meaning the probabilities are the same whether they are computed assuming the state is reduced at each step (on sequential 'measurement' of each projector $\widehat{P}_k$), or assuming it is always the uncollapsed state. Whether or not a set of histories is consistent depends on the basis used at each time, the state, and on the unitary dynamics.

Components of the state decohere, in some basis, when there exist effective, local equations of motion, propagating data along individual branches of the state (referred to that basis), in approximate agreement with (and usually adding corrections to) the results of the conventional formalism using the measurement postulates. It goes without saying that such histories are

approximately consistent – interference effects between histories could hardly be significant if there are such effective in-branch equations. In fact consistency is a much weaker requirement. One can define consistent histories for the simplest imaginable systems, with only a handful of degrees of freedom. One can represent the motion of a single electron in an inhomogeneous magnetic field (the Stern–Gerlach experiment) in accordance with various bases, each of them consistent; one can even smoothly modify one basis into another whilst maintaining consistency [6]. Non-interference by itself is not enough to guarantee that any interesting physics – effective equations – attaches to these histories. In the language of Gell-Mann and Hartle [7], decoherence proper concerns a *quasiclassical domain* and not just a consistent history space.

Finally, decoherence matters to the Everett interpretation. For this interpretation, like all the theories so far considered, there is a wave function of the universe, and like the consistent histories theory, the unitary dynamics alone is fundamental. As in all of these theories there is a preferred basis, defined by decoherence. We shall say more about the Everett interpretation shortly.

Defining probability in terms of decoherence, one has more or less the same Boolean sublattice of projectors throughout – a coarse-graining of projectors on configuration space (equivalently, on phase space). But with that it is clear that the premise of Gleason's theorem need not apply, indeed, that it has no motivation whatsoever. If probability only makes sense in the context of decoherence, which only arises for certain dynamical variables and in certain situations, why suppose that probabilities can be defined for arbitrary resolutions of the identity, with a non-contextual additivity requirement built in from the beginning? Why suppose probability has any meaning at all in regimes in which dynamical decoherence does not exist? But as we shall see, the Born rule can be derived from alternative premises, that from the point of view of decoherence theory are very natural.

12.4 A New Derivation of the Born Rule

If probabilities only arise in the context of decoherence – if they are 'emergent' – then they will have to have some of the key attributes of decoherence.

- The first is that decoherence typically involves highly degenerate, indeed infinitely degenerate, projectors (a projector onto any non-zero volume of configuration space must have infinite-dimensional range). Call this the *degeneracy* condition.
- The second is that decoherence is only *approximate*; there is no precise boundary below which the probabilities are undefined. Except at very low temperatures, there are many orders of magnitude over which projectors can be further fine-grained without loss of decoherence. Call this the *stability* condition.

- The third is that the probability rule should be basis-independent; it is the intrinsic relationship between the universal state and the Boolean lattice of projectors that matters. Call this the *invariance* condition.

Let us make this more precise, taking, for the sake of definiteness, for configuration space C, a family $F(C)$ of coarse-grainings Δ of C, with the natural partial ordering given by the inclusion relation on C. For each $\Delta \in F(C)$ one has an associated Boolean sublattice B_Δ (generated by projectors onto the cells of Δ) of the lattice of projectors on the total Hilbert space H. We are looking for a probability measure $\mu : H \times B_\Delta \to [0,1]$, $\Delta \in F(C)$ on projectors of infinite dimension which is intrinsic to H and stable under variation of Δ. So we require:

(i) Each $\widehat{P} \in B_\Delta$, $\Delta \in F(C)$, $\widehat{P}$ has infinite range (degeneracy).
(ii) For any unitary map $U : H \to H$,

$$\mu_\Delta(\psi, \widehat{P}) = \mu_\Delta(U\psi, U\widehat{P}U^{-1}) \quad \text{(invariance)}\,.$$

(iii) For any $\widehat{P} \in B_\Delta \cap B_{\Delta'}$, $\Delta', \Delta \in F(C)$,

$$\mu_\Delta(\psi, \widehat{P}) = \mu_{\Delta'}(\psi, \widehat{P}) \quad \text{(stability)}\,.$$

To these we shall eventually add a continuity assumption. Our claim is that under these assumptions, for sufficiently large families $F(C)$ of coarse-grainings of C, the Born rule follows uniquely. The proof in the case of regular polyhedra at every scale in C [denote $F_\infty(C)$] is particularly simple, although it has the disadvantage that it is unphysical; at sufficiently small scales decoherence inevitably fails. We shall later consider whether this idealization is really troublesome.

Certain results follow independently of any assumptions on $F(C)$. First a definition. Call a set of orthonormal vectors $\{\varphi_k\}$ *separating* for a set of disjoint projectors $\{\widehat{P}_k\}$, $k = 1, \ldots, d$, if $\widehat{P}_j\varphi_k = \delta_{jk}\varphi_k$. Then:

Lemma 1. *Let* $\{\varphi_k\}$, $\{\widehat{P}_k\}$, $k = 1, \ldots, d$, *be separating,* $\widehat{P}_k \in B_\Delta$, *and let* $\psi = \sum_{k=1}^d c_k\varphi_k$. *If* μ *satisfies (ii) and* $\psi' = \sum_{k=1}^d |c_k|\varphi_k$:

$$\mu_\Delta(\psi, \widehat{P}_j) = \mu_\Delta(\psi', \widehat{P}_j) \quad j = 1, \ldots, d\,.$$

Proof. Let $c_k = \exp(\mathrm{i}\theta_k)|c_k|$, $k = 1, \ldots, d$, and let $U_\theta : \varphi_k \to \exp(-\mathrm{i}\theta_k)\varphi_k$, $U_\theta \widehat{P}_k U_\theta^{-1} = \widehat{P}_k$, $k = 1, \ldots, d$ (such a U_θ can always be constructed); the result is immediate from (ii).

Likewise, the overall phase of the vector ψ is irrelevant for the probabilities. Now for the equiprobable case:

Lemma 2. *As in Lemma 1, but let* $|c_k|^2 = const$. *If* μ *satisfies (i), (ii):*

$$\mu_\Delta(\psi, \widehat{P}) = \begin{cases} 1/d\,, & \widehat{P} \in \{\widehat{P}_k\}\,, \quad k = 1, \ldots, d\,, \\ 0\,, \widehat{P} \text{ orthogonal to } \sum_{k=1}^d \widehat{P}_k\,. \end{cases}$$

Proof. By Lemma 1, we may assume without loss of generality that the c_ks are all real: $\psi = \text{const.}(\varphi_1 + \cdots + \varphi_d)$. First assume $d > 1$. Define U_π by $U_\pi \varphi_k = \varphi_{\pi(k)}$, $U_\pi \widehat{P}_k U_\pi^{-1} = \widehat{P}_{\pi(k)}$, where π is a permutation [such a U_π can always be constructed, since by (i) every projector has the same dimension]. Since $U_\pi \psi = \psi$, from invariance:

$$\mu_\Delta(\psi, \widehat{P}_j) = \mu_\Delta(\psi, \widehat{P}_{\pi(j)}) \, .$$

Choose any $\widehat{P}_k$ (say $k = 1$) and define $\widehat{P}'_1 = \widehat{P}_1 + (I - \sum_{j=1}^d \widehat{P}_j)$. Evidently $\{\varphi_j\}$ is separating for $\widehat{P}'_1, \widehat{P}_2, \ldots, \widehat{P}_d$ and by the same argument μ_Δ is constant on this set as well. Since $\widehat{P}'_1 + \widehat{P}_2 + \cdots + \widehat{P}_d = I$ and μ_Δ is a probability measure $\mu_\Delta(\psi, \widehat{P}_j) = 1/d$, $j = 2, \ldots, d$. The same argument for any other choice of k yields $\mu_\Delta(\psi, \widehat{P}_1) = 1/d$; from additivity again, if $\widehat{P}$ is any projector in B_Δ orthogonal to all the $\widehat{P}_k$s, then $\mu_\Delta(\psi, \widehat{P}) = 0$. The case $d = 1$ does strictly speaking involve an additional (but very weak) assumption: that there are at least 3 disjoint projectors in B_Δ disjoint from $\widehat{P}_1$, denoted $\widehat{P}_2, \widehat{P}_3, \widehat{P}_4$. Let $\widehat{P}'_4 = 1 - \sum_{k=1}^4 \widehat{P}_k$. Since by assumption $\widehat{P}_k \psi = 0$, $k = 2, 3, 4$, $\mu(\widehat{P}_2) = \mu(\widehat{P}_3) = \mu(\widehat{P}'_4)$, by the result already proved (using a permutation that leaves $\widehat{P}_1$ invariant). Let $\widehat{P}'_2 = \widehat{P}_2 + \widehat{P}_3$; again $\widehat{P}'_2 \psi = \widehat{P}'_4 \psi = 0$ so $\widehat{P}'_2$ and $\widehat{P}'_4$ are equiprobable. But then $\widehat{P}'_2$ and $\widehat{P}_2$ are equiprobable, and from additivity $\mu(\widehat{P}_2) = 0$. By the same argument $\mu(\widehat{P}_3) = \mu(\widehat{P}'_4) = 0$, so by additivity $\mu(\widehat{P}_1) = 1$.

The case $d = 1$ is the *eigenvector–eigenvalue rule*; this result and the method of proof follow closely the mathematical ideas introduced by Deutsch [5] and Wallace [20, 22]. The next two lemmas and Theorem 2 differ in certain respects, however. We shall come back to the Deutsch–Wallace theorem shortly.

We need the stability condition to go beyond the equiprobable case. The proof for $F(C) = F_\infty(C)$, where C is R^n and H is isomorphic to $L^2(R^n, \mathrm{d}x^n)$, is as follows. [Note that condition (i), degeneracy, is no longer needed as an independent assumption.]

Lemma 3. *For any $\psi \in H = L^2(R^n, \mathrm{d}x^n)$, $\Delta \in F_\infty(C)$, and for any $\widehat{P} \in B_\Delta$ and any integer m, there exists a refinement Δ' of Δ and orthogonal projectors $\widehat{P}_j \in B_{\Delta'}$, $j = 1, \ldots, m$, summing to $\widehat{P}$, such that $\left|\widehat{P}_j \psi\right|$ is constant.*

Proof. Let $\psi' = \widehat{P}\psi \neq 0$ (if zero, the proof is immediate). For $n = 1$, $\int_{-\infty}^r \overline{\psi}' \psi' \mathrm{d}x$ is a non-negative increasing function of r. By the intermediate value theorem, there are real numbers $r_1, \ldots, r_{m-1}$ such that $\int_{r_j}^{r_{j+1}} \overline{\psi}' \psi' \mathrm{d}x = \text{const.}/m$, $j = 0, \ldots, m-1$, $r_0 = -\infty$, $r_m = \infty$. Choose as projectors the characteristic functions $\chi_{\Delta'_j}$ on R, where $\Delta'_j = [r_j, r_{j+1}]$. The generalization to higher dimensions is obvious.

We may then prove

Lemma 4. *Let μ be a probability measure on $B_\Delta \in F_\infty(C)$ satisfying (ii) and (iii). Let $\{\varphi_k\}$, $\{\widehat{P}_j\}$, $k = 1, \ldots, d$ be separating. Let*

$$\psi = const. \sum_{k=1}^{d} \sqrt{m_k}\varphi_k \ , \quad m_k \in Z \ .$$

Then:

$$\mu_\Delta(\psi, \widehat{P}) = \begin{cases} \dfrac{m_j}{\sum_{k=1}^{d} m_k} \ , & \widehat{P} = \widehat{P}_j \ , \quad j = 1, \ldots, d \ , \\ 0 \ , & \widehat{P} \text{ orthogonal to } \sum_{k=1}^{d} \widehat{P}_k \ . \end{cases}$$

Proof. By Lemma 3, we may choose a fine-graining $\Delta' \in F_\infty(C)$ of Δ such that for each $k = 1, \ldots, d$, $B_{\Delta'}$ contains m_k orthogonal projectors $\widehat{P}_k^j$ summing to $\widehat{P}_k$, satisfying $\left|\widehat{P}_k^j \varphi_k\right| = \text{const}$. Define $\varphi_k^j = \widehat{P}_k^j \varphi_k$, then

$$\varphi_k = \frac{1}{\sqrt{m_j}} \sum_{j=1}^{m_k} \varphi_k^j \quad \text{and} \quad \psi = \text{const.} \sum_{k=1}^{d} \sum_{j=1}^{m_k} \varphi_k^j \ .$$

By construction $\{\varphi_k^j\}$ is separating for $\{\widehat{P}_k^j\}$ ($m = \sum_{k=1}^{d} m_k$ in all) and the conditions of Lemma 2 apply; so $\mu_\Delta(\psi, \widehat{P}_k^j) = 1/m$ and $\mu_\Delta(\psi, \widehat{P}_k) = \mu_\Delta(\psi, \sum_{j=1}^{m_k} \widehat{P}_k^j) = m_k/m$.

It is a short step to the general case. We need only assume that μ_Δ is continuous as a map $H \to [0, 1]$ (for fixed $\widehat{P} \in B_\Delta$). We thus obtain:

Theorem 2. *Let μ be as in Lemma 4 and for each $\widehat{P} \in B_\Delta$ let*

$$\mu_\Delta(\,\cdot\,, \widehat{P}) : H \to [0, 1]$$

be continuous in norm. Then for any $\psi \in H$:

$$\mu_\Delta(\psi, \widehat{P}) = \frac{\langle \psi, \widehat{P}\psi \rangle}{\langle \psi, \psi \rangle} \ .$$

The proof proceeds by constructing, for any ψ, a sequence of vectors in H of the form assumed in Lemma 4 that is separating for $\widehat{P}$ and $I - \widehat{P}$ that converges to ψ.

Having stated the theorem, two caveats. The first is that since it assumes continuity in norm, its mathematical interest is considerably diminished. One of the remarkable things about Gleason's theorem is that continuity of the measure is *derived*. But from a physical point of view, if probabilities depend at all on vectors in H, they surely vary continuously with them. The assumption is physically perfectly natural.

The second is the one already noted, that $F_\infty(C)$ is unphysical. But we should be clear why the idealization was needed. It is because, in Lemma 3, the integers m_k arising may be arbitrarily large, so the number of orthogonal projectors required to sum to each $\widehat{P}_k$ must be arbitrarily large. That is only possible if we allow coarse-grainings of C that are arbitrarily small.

Suppose the scale of the coarse-graining is bounded below; what sort of restriction does this place on these numbers? We are only interested in exploring the probabilistic structure of the state at the decoherence length scale and above (for probability, if emergent, has no meaning at smaller length scales). So we may suppose the state is approximately uniform over some region Δ_k of configuration space, near the threshold of decoherence, at the length scale $2l$; let Δ' be a refinement of Δ at the length scale l; how many disjoint projectors in $B_{\Delta'}$ are there, summing to the projector on Δ_k? The answer, for configuration space of dimension n, in the case of hypercubes, is 2^n. So even taking the limits of decoherence into account, we can bound real-valued ratios of probabilities above and below by rational ratios of enormously large integers – numbers which increase exponentially with the number of degrees of freedom. Given our general philosophy, that probabilities are only defined given decoherence and that they should be robust under changes of coarse-graining, we can legitimately demand that the distribution $\mu_\Delta(\psi)$ should be smooth and not only continuous under variations in ψ, at least for macroscopic systems of large numbers of degrees of freedom. It should be effectively constant over variations in ratios of norms of one part in $2^{10^{22}}$.

We come back to the dependence of the Born rule on the *purpose* of the experiment. Although we do not yet have a clear picture of how to interpret quantum mechanics using decoherence theory, we have an unambiguous answer to this question. It is 'design' and not 'purpose'; it is a matter of what, at the sub-decoherence level, as described in pure quantum mechanics, is reliably correlated with decohering variables. It is only by virtue of these correlations that probability as emerging with decoherence has any meaning at the microscopic level.

Here the details are familiar; they follow, in the simplest cases, the von Neumann treatment of measurement processes. The measurement interaction brings about correlations between projectors in B_Δ, $\Delta \in F(C)$, with projectors onto eigenspaces of dynamical variables of individual subsystems $a, b, c, \ldots$, described by (possibly finite-dimensional) subspaces $H^a \subset H$. The only limit to this process is the ingenuity of the experimenter. In the case of spin systems of small dimension, it is a plausible claim that in this way one can experimentally realize correlations between projectors in B_Δ and *arbitrary* projectors on H^a.

So long as projectors in B_Δ and on H^a can be reliably correlated in this way – depending on the ingenuity of the experimenter – probabilities assigned to projectors in B_Δ can be assigned to projectors on H^a as well.

That is the whole story about probability at the sub-decoherence level. That these correlations are non-contextual follows automatically.

12.5 The Everett Interpretation

Probability, if only defined in the context of decoherence, must be given by the Born rule. But what is the underlying physical picture? We have spoken of quantum mechanical models of the experimental apparatus, applying quantum mechanics directly to macroscopic systems, but of course decoherence theory does not in itself solve the conceptual problems that follow from this. Lack of clarity on this score makes it hard to answer the questions we are concerned with:

1. What is objective probability?
2. Why should subjective expectations track these objective probabilities?

If we want clarity as to questions of what exists, we had better look to a realist solution to the problem of measurement. If we want probability to arise only in the context of decoherence, we had better not modify or add new elements to the unitary formalism. This narrows down the available alternatives. There are versions of the consistent histories interpretation that may lay claim to this status, but those in which only a single history is realized necessarily forsake the approximate character of decoherence (essential to the derivation of the Born rule that we have given), and require instead some new input to the theory in order to single out a unique history space (to which the one and only history actually realized belongs). The idea of environment superselection rules and the interpretation of an improper mixture (arrived at by tracing out environmental degrees of freedom) in terms of ignorance has been dropped even by its advocates [24, 25].

That leaves only the literalist interpretation of the state, in which all the branches are physically real. With that we are led to many worlds and to the Everett interpretation: worlds are described by the components of the universal state referred to the decoherence basis. As such, under the unitary dynamics, the evolution from a single component of this basis into a superposition is the evolution of one world into many. Worlds in this sense divide.[1] A chance process is one in which a system is subject to division in this sense.

12.5.1 Understanding Branching

Our objective here is not to evaluate solutions to the measurement problem, only the status of probability within them. In Everett's approach, there

[1] They may also, in principle, recombine. It has long been recognized that probability and the arrow of time are intimately related. This relation leads on to others [23]; we cannot do justice to them here.

is now a clear-cut answer to (1): probabilistic events arise only by branching. Branching, or equivalently, the development of a superposition (referred to the decoherence basis), is the basis of all objective physical indeterminism (for quantum mechanics is taken to be both universal and fundamental). The moment of branching is, to use Heisenberg's language, the point at which 'potentiality' becomes 'actuality'. Chances, as quantities, are squares of the norms of the associated transition amplitudes – all categorical physical properties and relations.

Just as important, branching (and therefore this transition) inherits the approximate character of decoherence. One can put this in terms of *vagueness* – that branching is vague, with clear-cut instances but no sharp boundaries. Vagueness permeates ordinary language, but it is pervasive in scientific theories as well. There is no precise physical definition of tables or chairs, no more than of cells or molecules. Vagueness is endemic in the chain of reduction, from ordinary objects to material science, the solid state, and chemistry; from zoology and anatomy to molecular biology and biochemistry. According to the Everett interpretation, extracting quasiclassical phenomenology from the unitary dynamics of quantum mechanics is subject to the same kinds of equivocation as confront any program for recovering higher-level laws from more fundamental ones [21]. The methodological issues are all precisely the same.

If chances arise with branching, but branching depends on the details of the coarse-graining, then chances can only be stable under variations in coarse-graining if they satisfy (iii), and hence [with assumptions (i), (ii)] the Born rule – this is the result just proved. It replaces Gleason's theorem; probability is not assumed from the outset to be non-contextual and defined for any basis; it is not assumed to have any fundamental significance at all. Probability is 'emergent'.

One might object that the answer to question (1) is then not so clear-cut after all; chances arise with branching, but branching, because imprecisely defined, is hardly being accounted for by any precise properties and relations. But the same is true of paradigm cases of successful inter-theory reduction. Reduction is never precise. It is not as though there should be some precise and unique frequency distribution in electromagnetism that corresponds to the colour 'red', for example. The point about the reduction in the case of chance is that it be to *categorical* properties and relations (that are not themselves indeterminate, borderline, or chancy); it is that the substrate posited by the reducing theory (the spectrum of waves) should not employ concepts just as mirky as those we sought to elucidate (the colour red). This is where Popper went wrong with his account of chance in terms of dispositions.

In fact, in the special case of laboratory experiments (or more generally of 'interpreted' phenomena), a more abstract notion of branching is available that is reasonably precise; here one defines branches, by sheer stipulation, in $1:1$ correspondence with the different numbers assigned to measurement

outcomes (as equivalence classes of configurations of the experimental display, that are all taken to represent the same numerical outcome). The number of branches is the number of possible pointer positions on the dial. Of course there still remain problems of borderline cases, if for no other reason than that an experiment is always subject to inefficiencies and is always prone to malfunction; but at this level, concerning branches that we count as differing in clear-cut respects, we will be down to a small and finite number. We will be perfectly able to make sense of their number.

The Everett interpretation does well with (1). It does much better than the pilot-wave theory, even though the latter has all the resources of the Everett interpretation – and then some, for it postulates additional structure, namely a particular trajectory. But that is just where the trouble comes in (when one introduces the trajectory); the trajectory may be one in which the statistics are completely different from those predicted by the Born rule. What does probability mean in such a case?

Some have thought that precisely the same worry arises in the Everett interpretation. There too there exist 'anomalous branches', in which the recorded statistics do not match the ones predicted by the Born rule. But there is an important difference. According to Everett, there is nothing about a branch of this kind that can ensure it will *continue* to violate the Born rule (for there is no fact of the matter as to what will happen following on from a given branch, so long as every branch is given to division), unlike the situation for anomalous trajectories in pilot-wave theory. Anomalous branches, in the Everett interpretation, are like statistically anomalous segments, each of finite length, in an infinitely extendible sequence of random numbers. They have to exist if the sequence is to be genuinely random, but in no sense is any given subsequence likely to continue to be anomalous.

What about (2), the connection with subjective expectations? Why should the amplitudes on branching be our guide for these? But at least it is clear that on branching we *ought* to be concerned with weights for branches. For it is obvious that branching – *personal* branching, literally dividing in two, say – will lead to divided expectations, and this will be so even given *complete* knowledge of the branching process. The two successors may differ widely, yet each will with as much right call themselves the same person as before. In the face of branching there is no 1 : 1 criterion of identity in the forward direction of time. But if one is to make provision for one's successors, one must allocate resources among them. And one can hardly do this without introducing weightings, implicit or explicit, in one's reasoning. One cannot ready oneself for anything and everything.

Philosophers have long disagreed on how, in the presence of branching, questions of personal identity are to be settled [14]; we should make no pretence that in this matter there is any real consensus. But the one response that is really damaging to the Everett interpretation has found few advocates: it is that in the face of branching one should expect *nothing*, *oblivion*.

The view is inherently implausible, given that each successor is *ex hypothesi* functionally exactly the same as me. Every successor has all of my attributes and memories; every successor professes himself to be me on the basis of physical continuity (and on every other physical criterion). No wonder the view has found few supporters.

12.5.2 Deutsch's Argument

If not oblivion, then divided expectations. If divided expectations, then divided how, and with what weighting? What preferences ought one to have for one process of branching (for performing one choice of experiment), with a given utility in each branch (for each experimental outcome), over another? But just at this point Deutsch's argument comes into play.

Deutsch's strategy, following de Finnetti [4], and before him von Neumann and Morgenstern [19], was to define subjective probabilities (hereafter, *weights*) in terms of the preferences of a rational agent among a set of games $g \in G$ each with some set of outcomes E_k, $k = 1, \ldots, d$, with associated utilities ('payoffs') – concrete rewards, cash prizes say, that the agent values – belonging to some set $\mathcal{U}$. Call $\mathcal{P} : E_k \to \mathcal{U}$ the *payoff function* for that agent. If rational, the ordering $\preceq$ on G defined by one's preferences should satisfy certain obvious rules (for example, transitivity). The strategy is then to find a strong enough but still plausible set of rules sufficient to ensure that for each game g there exists real numbers $p_k \in [0, 1]$ for each outcome λ_k, $k = 1, \ldots, d$, summing to one, and quantities $\mathcal{V}(g) = \sum_{k=1}^{d} p_k \mathcal{P}(\mathcal{E}_k)$, such that $g \preceq g'$ if and only if $\mathcal{V}(g) \leq \mathcal{V}(g')$. If G is big enough, indeed, one would hope to show that the numbers p_k (weights) for the outcomes in each g are unique. The important point in this is that the p_ks arrived at in this way will be *independent of* an agent's utilities. A rational agent will act as if attempting to maximize the expectation value of her utilities, using these weights as probabilities. It is because of this representation theorem that subjective probability is in so much better shape than objective probability. If this is what probability is, one can explain why it obeys the rules that it does.

Deutsch's remarkable claim is now that the preference ordering of rational agents, in the face of quantum games, can be so constrained that the weights defined by these preferences (independent of their actual utilities) agree with the Born rule.

This result is so surprising that one wants to have an inkling of how it was obtained. Here we shall follow Wallace [20], who has substantially revised and simplified the argument. A quantum game can be played using any quantum experiment, simply by agreeing on various payoffs (positive or negative) on each possible outcome. What is an experiment, according to Everett? It is a special kind of process, involving stable macroscopic objects, described by effective equations, such that states can be attributed to subsystems (typically molecular), as relative states, that evolve unitarily (i.e., as

a product state) with respect to the apparatus (this is the state-preparation process). Following some unitary evolution preserving this product structure, they evolve into an entanglement with the measurement device. Components of this eventually include macroscopic degrees of freedom (pointer positions).

From a mathematical point of view one introduces a tensor product in the Hilbert space for a particular branch, distinguishing some microscopic subsystem a with Hilbert space H^a from all the rest. Suppose (for simplicity) that H^a has finite dimensions. The state preparation device produces, in a reliable way, vectors in a certain subspace of H^a (for simplicity suppose 1-dimensional, so a particular state ϕ), in a tensor product with the state of the rest of the apparatus and its environment. The entanglement subsequently brought about is with some set of orthogonal states $\phi_k \in H^a$, $k = 1, \ldots, d$, of a, with decohering states of the apparatus and environment (grouped together when they give rise to the same pointer reading). The latter reliably leave behind them a macroscopic trace.

In these models it is useful to introduce numbers λ_k for the states in H^a which have some dynamical significance – eigenvalues λ_k, associated with eigenstates of some self-adjoint operator $\widehat{X} = \sum_{k=1}^{d} \lambda_k \widehat{P}_{\phi_k}$; these replace the E_ks above. The instrument display, meanwhile, registers numbers concretely, so one has some definite assignment of the λ_ks with these numerals (usually taken as the identity). The experiment is converted to a game by specifying a map from these numerals to an agent's utilities in $\mathcal{U}$, which we can model directly in terms of the payoff function as a map $\mathcal{P} : Sp(\widehat{X}) \to \mathcal{U}$. Suppressing explicit reference to H^a, a quantum game is then given by an ordered triple $\langle \phi, \widehat{X}, \mathcal{P} \rangle$.

But adopting this schema, we must recognize the arbitrary elements in it. It is obviously possible to compensate for a change in labels λ_k by a change in the payoff function. This corresponds to a certain arbitrariness in the choice of self-adjoint operator that the experiment is said to measure: whether it is $\widehat{X}$ with payoff function $\mathcal{P}$, or $f(\widehat{X})$ with payoff function $\mathcal{P} \circ f^{-1}$ [for some invertible f on $Sp(\widehat{X})$]. For another arbitrary element, typically the initial product state involving ϕ will be subject to a unitary evolution U on H^a which preserves the product state. Indeed, the preparation device may best be modelled using a sequence of Hilbert spaces with intertwining operators $U : H^a \to H^b$. From an Everettian standpoint it is now entirely arbitrary which of these is taken to be *the* initial state; the experiment can with as much right be called a measurement of $U\widehat{X}U^{-1}$ in the state $U\phi$ as of $\widehat{X}$ in the state ϕ. There is nothing in the physics to say. It is only the correlations between vectors in H^a under the unitary dynamics, in a definite time relation with the event of a pointer reading or payoff in $\mathcal{U}$, that are relevant to an experiment; precisely which state is called the initial one is immaterial.

We have established two principles. Under our schema for quantum games, the triples $\langle \phi, \widehat{X}, \mathcal{P} \rangle$, many games can be realized by a single physical process. Since preferences among games should concern the physical world rather than

the models used to describe it, they should value games as the same if they can be realized by the same physical process. Let $g \sim g'$ if and only if $g \preceq g'$ and $g' \preceq g$. We require for any unitary on H^a and any invertible f on $Sp(\widehat{X})$ the equivalence principles:

Payoff Equivalence

$$\langle \phi, \widehat{X}, \mathcal{P} \rangle \sim \langle \phi, f(\widehat{X}), \mathcal{P} \circ f^{-1} \rangle \ .$$

Measurement Equivalence

$$\langle \phi, \widehat{X}, \mathcal{P} \rangle \sim \langle U\phi, U\widehat{X}U^{-1}, \mathcal{P} \rangle \ .$$

This is a schema well-suited to the Everett interpretation, but it can be motivated on other grounds; it can even be motivated operationally [15]. Deutsch's decision-theoretic axioms naturally make no reference to the Everett interpretation. The analysis that follows can, therefore, be largely freed from its dependence on Everett. But as we shall see, it then fails to have the foundational significance for probability that we are after. We shall come back to this point in due course.

First Deutsch's decision-theory axioms. We will prove only one of his results, and for that we only need two axioms. For simplicity, we assume that $\mathcal{P}$ is linear [so $\mathcal{P}(x_1 + x_2) = \mathcal{P}(x_1) + \mathcal{P}(x_2)$ – this is a convention on the labels λ_k]. Let $f_s : R \to R$ be the function $f_s(x) = x + s$, and let $-I : R \to R$ be $-I(x) = -x$. The first axiom is:

Sure-Thing Principle

$$\text{Let } g = \langle \phi, \widehat{X}, \mathcal{P} \rangle \ , \ \ g' = \langle \phi, \widehat{X}, \mathcal{P} \circ f_s \rangle \ ; \ \text{then } \mathcal{V}(g') = \mathcal{V}(g) + \mathcal{P}(s) \ .$$

I am indifferent between receiving $\mathcal{P}(s)$, and then playing game g, and playing g and then receiving $\mathcal{P}(s)$, whatever the outcome. But the latter is g'.

The second axiom is:

Zero-Sum Rule

$$\text{Let } g = \langle \phi, \widehat{X}, \mathcal{P} \rangle \ , \ \ g' = \langle \phi, \widehat{X}, \mathcal{P} \circ -I \rangle \ ; \ \text{then } \mathcal{V}(g) = -\mathcal{V}(g') \ .$$

It must be possible for me and my banker to share exactly the same preferences, and to play the same game: what I am prepared to pay to play g, I pay to him. The most I am prepared to pay should be the least he is prepared to accept. But whereas I play g, he plays g'.

The rationale for these principles can also be stated in a way that takes branching explicitly into account. For the first, if I accept $\mathcal{P}(s)$ before playing g, each of my successors inherits $\mathcal{P}(s)$ as well (for the utility too is subject to branching), and the situation at the end is the same as if I had played g'. For

the second, if I am prepared to swap with my banker before playing g, being paid what I would otherwise have paid, each of my successors is swapped with my banker's successors, and pays what he would otherwise have been paid; but the latter is just g'.

With that it follows that for $\widehat{X} = x_1\widehat{P}_{\phi_1} + x_2\widehat{P}_{\phi_2}$,

$$\mathcal{V}(\phi_1 + \phi_2, \widehat{X}, \mathcal{P}) = \frac{1}{2}\mathcal{P}(x_1) + \mathcal{P}(x_2) \ ,$$

in accordance with the Born rule. For $\phi_1 + \phi_2$ is invariant under the permutation π of ϕ_1 with ϕ_2, so by payoff equivalence

$$\mathcal{V}(\phi_1 + \phi_2, \widehat{X}, \mathcal{P}) = \mathcal{V}(\phi_1 + \phi_2, U_\pi \widehat{X} U_\pi^{-1}, \mathcal{P}) \ .$$

By measurement equivalence this is $\mathcal{V}(\phi_1 + \phi_2, \widehat{X}, \mathcal{P} \circ \pi^{-1})$. Since

$$\pi^{-1} = \pi = -I \circ f_{-x_1 - x_2} \ ,$$

by the sure-thing principle and the linearity of $\mathcal{P}$, one obtains

$$\mathcal{V}(\phi_1 + \phi_2, \widehat{X}, \mathcal{P} \circ - I) + \mathcal{P}(-x_1 - x_2) \ .$$

By the zero-sum rule and linearity again, this is

$$\mathcal{V}(\phi_1 + \phi_2, \widehat{X}, \mathcal{P} \circ - I) = -\mathcal{V}(\phi_1 + \phi_2, \widehat{X}, \mathcal{P}) \ .$$

So

$$\mathcal{V}(\phi_1 + \phi_2, \widehat{X}, \mathcal{P}) = -\mathcal{V}(\phi_1 + \phi_2, \widehat{X}, \mathcal{P}) + \mathcal{P}(x_1) + \mathcal{P}(x_2) \ .$$

Deutsch called this his 'pivotal result', and for good reason: it is the first time that any rational basis has been found to tailor one's subjective probabilities to the quantum mechanical ones. It is also the first step towards proving a general principle. Observe that the argument goes through for any $\phi \in H$ of the form $\phi_1 + \phi_2 + c\phi_3$, where $\widehat{P}_1\phi_3 = \widehat{P}_2\phi_3 = 0$; observe further that the antecedent can be stated as the condition that the Born rule for $x_1\widehat{P}_{\phi_1}$ yields the same value as for $x_2\widehat{P}_{\phi_2}$. So we have proved that for any orthogonal projectors:

Special Equivalence

$$\text{If } \mu(\phi, \widehat{P}_1) = \mu(\phi, \widehat{P}_2) \ , \text{ then } \langle \phi, \widehat{P}_1, \mathcal{P} \rangle \sim \langle \phi, \widehat{P}_2, \mathcal{P} \rangle \ . \tag{12.2}$$

Further axioms of decision theory are required to derive the analogous condition in which the vectors in H^a are different but the $\widehat{P}$s are the same. Combining the two, we have [22]:

General Equivalence

$$\text{If } \mu(\phi_1, \widehat{P}_1) = \mu(\phi_2, \widehat{P}_2)\,, \text{ then } \langle \phi_1, \widehat{P}_1, \mathcal{P}\rangle \sim \langle \phi_2, \widehat{P}_2, \mathcal{P}\rangle\,.$$

Given the general equivalence condition, the full representation theorem follows from decision-theoretic axioms that are exceedingly weak – axioms which, in point of fact, should be acceptable whatever one's views on what it is proper to believe in the face of personal division [22]. This full representation theorem is then none other than the Principal Principle (as is the general equivalence rule for equiprobability). But there is an important difference, that the principle has been derived even under the condition that the agent has perfect knowledge. So it holds unrestrictedly; there is no room for the rider to the principle that a rational agent should be indifferent between playing two games if the objective probabilities for the same utilities are the same, *whatever* additional information she has, provided it does not bear on the actual outcomes of the games (a rider that has in fact proved notoriously difficult to state with any great precision). There is no need to exclude information of this sort because, of course, she knows everything there is to know about the outcomes of the games. As Wallace has stressed [22], an unrestricted principal principle cannot possibly be accepted, let alone deduced, on any interpretation of quantum mechanics in which only a single history is real; for why *should* one be indifferent between two games, even if they have the same probability as given by the Born rule, if you know their actual outcomes as well; will it not depend on what those outcomes are?

12.5.3 Measurement Neutrality

The general equivalence condition can in fact be stated in a way that is independent of any particular reference to experiments and independent of the schema we have used. As Wallace has shown, the derivation of the full representation theorem is all the simpler. Given general equivalence, the representation theorem is altogether unproblematic; the decision theory axioms are so weak that there is no need to consider what it is proper to believe on personal division. And it may be, on the basis of the Everett interpretation, that one can argue for the general equivalence condition directly.

But there is something to be said for the argument we have just sketched. We should hold fast to the belief that on fission, we should not anticipate oblivion; that, as argued by Parfit, psychological continuity is what matters, and not a relation with the formal properties of identity [12]. We should look forward to the same sort of first-person perspective, whatever it is, that we do in the absence of branching. But then our situation is one of subjective uncertainty, in the following sense: there is no *one* perspective that we should look forward to; we must, in some sense, entertain them all, and we must make provision for them all (or as many as we can reasonably survey), weighting them appropriately – just as we do in the face of uncertainty. It is only in

special situations – for example, when no successor has some outcome – that one is entitled to ignore that outcome completely, and give it zero weight.

These questions lead off into philosophy. The other sort of criticism that can be made of the Deutsch–Wallace derivation of the general equivalence condition (and the special equivalence condition as above) concerns the use of the schema for quantum games. Is this schematization (subject to payoff and measurement equivalence) sufficiently detailed? Does it tell you everything you really want to know in the context of decisions about real-life experiments?

According to Everett, we certainly can characterize quantum games by triples of the form $\langle \phi, \widehat{X}, \mathcal{P} \rangle$ (although that is common ground to a wide variety of approaches to foundations). Likewise the Everett interpretation licences payoff equivalence and measurement equivalence (although as mentioned, so can other approaches). But why should a rational agent take absolutely nothing else into account in determining her preferences? Might there not be other features of quantum measurements (or quantum games) that are worth taking into account?

One can simply deny this possibility. A principle to this effect has been called *measurement neutrality* [20]; it is the suspicion that measurement neutrality is too strong, or rests on unwarranted or incoherent assumptions as to what it is right to believe in the face of personal division, that prompts one to seek a more direct argument for the general equivalence condition. But better is to seek a direct argument for measurement neutrality.

Why believe in this principle? What more can be said of a measurement process, according to the Everett interpretation, not captured in the schema, and what is the rationale for ignoring it? There is of course a vast amount of information that is not contained in the schema. There is everything else happening in each branch; there is the world outside the laboratory (assuming games are played in laboratories), and there are all the detailed goings-on in the laboratory that were unmentioned in the payoff (including molecular goings-on that are not in fact detected). But if these are deemed to matter to her, let them be put into her utilities, and let her prize those games whose payoff functions include them explicitly. Still, if she is rational, her preferences will be consistent with the Born rule, along the lines of the argument just given. The real issue, it is now becoming clear, is that the one thing that she cannot put into her payoff function is the amplitude; it is not a possible payoff for her that can be arranged. For there is no dynamical process, according to Everett, whereby the amplitude of a branch can be measured, and its value exhibited by a display, or otherwise encoded into the payoff function $\mathcal{P}$. *No* unitary evolution can ever achieve that.

This goes a long way to explaining why the amplitudes should play the part in our mental lives that they do, when it comes to our preferences (appearing, distinctively, in the weights with which we view outcomes, rather than as part of the outcomes), and it makes clearer the generality of our

schema (that it can in principle be applied to arbitrarily complex initial conditions and payoff functions). But it will not do to explain why nothing else but amplitudes can matter to these weights. It is one thing to care about details of the apparatus as goes the physical outcomes produced by it – these should simply be entered into one's utilities. But what about details of the apparatus that effect its dynamical functioning, that are not captured in our simple schema?

There are of course a vast range of physical considerations bearing on the detailed dynamics of an experiment, but it is reasonable to distinguish between those that make a difference to the branching structure and those that do not. We are speaking of a rational agent who may know everything there is to know: for her, subjective probability, weights, arise in the first instance on her personal division.

Any dynamics not involving branching concerns the purely deterministic development of the branch (insofar as branches can ever evolve deterministically). Experiments modelled in the same way by our schema that differ on this are likely to have different efficiencies and will differ in their systematic errors and the ways they are prone to malfunction. There is no reason why rational agents should regard them as precisely equivalent, but equally, just because they concern deterministic effective processes, they seem unrelated to the foundational questions about probability.

It is otherwise with distinctions among experiments that turn on differences in branching. Again, they break down into two kinds. The first concerns the precise details whereby an initial entanglement of the microscopic system with the measuring apparatus is produced, including the branching that takes place as progressively larger numbers of degrees of freedom are entangled with it. This takes place over decoherence time scales and is extremely rapid. The second kind concerns subsequent branching unrelated to the coupling of the apparatus to the microscopic system, usually considered in terms of statistical fluctuations (and that go on continuously whether or not any measurement is performed). Of course experiments attempt to control the latter – this is noise in the signal that is better or worse eliminated – but in no sense is it possible to model this kind of branching explicitly. It is dealt with again at the level of effective equations.

We are left with the key arena in which the initial entanglement is established and subsequently propagated to include large numbers of degrees of freedom – the business of measurement theory proper. Is it not reasonable that a rational agent treat differently experiments that differ in these respects, even if they are modelled in the same way in our schema of quantum games? This is the key objection to the derivation of the general equivalence principle from decision-theoretic principles: the amplitudes (and weights as given by the Born rule) may be the same, but the branching structure introduced by the measurement may differ wildly.

12.5.4 Decoherence, Again

The objection that branch number may have a role in dictating preferences has to be understood in the right way. It is not that the number of outcomes counted as distinct when it comes to the payoff function – equivalently, the number of gradations on the instrument readout – may not matter to our preferences: that number is already explicit in our schema (in the specification of the triples $\langle \phi, \widehat{X}, \mathcal{P} \rangle$). It is hard-wired in the instrument display. As such it is available to a rational agent, to be incorporated in her utilities if she will. So she may favour quantum games with five outcomes rather than four, because five is her favourite number; or she may dislike outcomes with thirteen in the display. Anything physically realized in any branch can always be entered into her utilities, and be looked after at the level of her payoff function, without compromising her conduct as a rational agent – and therefore in accordance with the Born rule. She will act irrationally, however, if she believes her liking for fives is a reason to *weight* outcomes of games with five in the display as greater than those without, or to set the *weight* of any outcome with thirteen in it to zero. If she does this, she will have to violate the sure-thing principle or the zero-sum rule in some cases; or to hold the consequences of the Everett interpretation for payoff equivalence and measurement equivalence to be false. Exactly the same applies if she assumes that each outcome, corresponding to each gradation of the display, should have equal weight: she will be convicted of irrationality or ignorance of quantum mechanics.

The objection is not that. It is that the schema for measurements is leaving something out of account, not that what it does contain can be acted on irrationally. It is that for each outcome with each given payoff, the *number* of branches all with that outcome has been ignored. Like amplitudes, and unlike the number of gradations on the instrument display, this is not information that can be factored into one's utilities; these are not numbers that can be reliably realized in a branch by any unitary dynamics. All the more reason, then, to think that they may be relevant to agent preferences in the way that amplitudes are so relevant to her weights.

But the answer to this should now be obvious. *There is no such thing as this number*. The only significance it has in concrete physical terms is what is coded up in the number of instrument gradations. It is true that one can specify such a number theoretically, for a given choice of decoherence basis, but it has no categorical physical significance; it is not part of what is really there.

The reason no other number can be defined has already been rehearsed; it is because decoherence is an imprecise concept. Formally, the number of decohering branches corresponds to the number of decohering projectors – and, one has to add, "for a particular choice of coarse-graining on configuration space". There is no such thing as the finest decohering set of projectors. The picture, in Everett theory, of the wave function of the universe as an endlessly

branching tree, breaks up as one goes into the fine detail.[2] It is no criticism of the Deutsch–Wallace argument that it leaves out of account a physically meaningless quantity. The same applies to the branching that takes place constantly, independent of the measurement of the microscopic system *per se* ('background noise'); there is no such thing as the number of branches produced in this way either. It is no part of any principle of rationality to take note of what is not there.

We have come full circle. A rational agent, who knows everything there is to know about the physical world, will still have preferences among quantum games, and she ought to order her preferences consistently. In so doing (for a sufficiently rich set of games) she will act as if assigning a unique set of weights to outcomes (independent of the utilities that she assigns to them) that have to obey the rules of probability theory. If she believes quantum mechanics to be true, under the Everett interpretation, she will consider the schema for quantum games a reasonable idealisation of what goes on in measurements, subject to outcome equivalence and measurement equivalence. Moreover, she ought to believe in the sure-thing principle and the zero-sum rule (although here there are weaker principles that will do as well), so she ought to conclude from the equality of the norms of amplitudes for outcomes to equality of the weights that she gives to them. She ought to believe in the special equivalence condition. And so on, to the general condition and the full representation theorem. But the reason she should consider the schema for experiments adequate (although she ought to have quibbles over the neglect of detector inefficiencies and the presence of background noise) is because her subjective weightings depend on branching, and branching depends on decoherence; it is because of what the chances are, in physical terms, that there is no fact of the matter as to the number of branches. And, conversely, it was exactly because decoherence is a matter of approximation, that if chance is to emerge with decoherence, then it had better be stable under changes in coarse-graining – equivalently, under change in branch number – that we were able to isolate the ratios in modulus squares of the amplitudes (for decohering projectors) as the only invariant quantities that could play the role of chance.

Subjective and objective probability emerge at the end of the day as seamlessly interjoined: nothing like this was ever achieved in classical physics. Philosophically it is unprecedented; it will be of interest to philosophers even if quantum mechanics turns out to be false, and the Everett interpretation consigned to physical irrelevance; for the philosophical difficulty with probability has always been to find *any* conception of what chances are, in physical terms, that makes sense of the role that they play in our rational lives.

Still, the Everett interpretation is inherently fantastic; one would like if possible to free the argument from any dependence on it. Yet we encountered again and again points on which the Everett interpretation played a critical

[2] It is therefore welcome that the Everett interpretation can be formulated purely in terms of history spaces and the concept of a quasiclassical domain [16].

role – where the very features of the approach that make it unbelievable were of special salience. Introduce additional elements, over and above the unitary dynamics – whether hidden variables or an additional stochastic dynamics controlling the state – and the symmetries used to derive the proofs for the individual case have to be broken. But how else to solve the problem of measurement if there is only a single world? Try to reformulate the derivation so as to apply to probability distributions over ensembles, and we are back to the same foundational questions about the latter as in classical theory. And meanwhile the very conclusions of the argument become obviously untenable: the unrestricted special equivalence condition that we derived is incoherent if there is only a single history. The arguments we have considered give no hope at all that one can derive the principal principle on any basis but Everett's.

It is ironic that the interpretation of probability in the Everett interpretation has always been thought to be its weakest link. On the contrary, it seems that it is one of the strongest points in its favour.

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13 On Hamilton–Jacobi Theory as a Classical Root of Quantum Theory

Jeremy Butterfield

> *"Don't worry, young man: in mathematics, none of us really understands any idea – we just get used to them."*
> John von Neumann, after explaining (no doubt very quickly!) the method of characteristics (i.e., Hamilton–Jacobi theory) to a young physicist, as a way to solve his problem; to which the physicist had replied: "Thank you very much; but I'm afraid I still don't understand this method."

This chapter gives a technically elementary treatment of some aspects of Hamilton–Jacobi theory, especially in relation to the calculus of variations. The second half of the paper describes the application to geometric optics, the optico-mechanical analogy and the transition to quantum mechanics. Finally, I report recent work of Holland providing a Hamiltonian formulation of the pilot-wave theory.

13.1 Introduction

In the eighty years since its discovery in the mid-1920s, quantum mechanics has gone from strength to strength. It has repeatedly been proved successful, to a high degree of accuracy, in domains of application very different from its original one. For example, although it was devised for systems of atomic dimensions (10^{-8} cm), it has since proven accurate for scales much smaller (cf. the nuclear radius of ca. 10^{-12} cm) and vastly larger (cf. superconductivity and superfluidity, involving scales up to 10^{-1} cm). Similarly, if we think of domains of application, not as length (or energy) scales, but as types of 'stuff' to which the theory applies. Though quantum mechanics was first devised to apply to matter (i.e., electrons and protons, the more 'obvious' constituents of atoms), it was soon extended to fields, i.e., the electromagnetic field: indeed, matter soon became regarded as excitations in associated fields. Similarly, if we think of domains of application as types of force: though first devised for electromagnetic forces, quantum mechanics now successfully describes the weak and strong forces. Indeed, similarly for 'domains' understood naively, as regions of the universe: quantum mechanics has also been applied with great success to astronomy – the obvious examples being the use of nuclear physics in theories of stellar structure and evolution, and of particle physics in theories of the early universe.

So quantum mechanics has been an amazing success story. I stress this point at the outset, for two reasons. First, it is, unfortunately, all too easy to get used to success. Nowadays, both physicists, for whom the various quantum theories have become everyday professional tools, and the wider scientifically literate public, can easily lose their sense of wonder at this immense success. So it is worth remembering how contingent, and surprising, it is.

My second reason is more specific to work in the foundations and/or philosophy of quantum theory. This work focusses on the interpretative problems, especially the measurement problem, that still confront quantum mechanics, despite its immense empirical success: hence this volume's question: Quo vadis, quantum mechanics? Of course, I endorse that focus: it is crucially important to address these problems. But in addressing them, it is salutary to recall this success, as an intellectual backdrop. Indeed, not only is it salutary: it might also be heuristically useful – though of course, different researchers, with their different intellectual temperaments, will take this success to give different heuristic clues about 'Quo vadis, quantum mechanics?'. For example, an Everettian philosopher such as Saunders (see Chap. 12) may see the success of the established quantum theoretic formalisms as supporting their position: certainly, heterodox quantum theories such as dynamical models of wave function collapse face an enormous task in recovering that success. On the other hand, a theoretical physicist who is searching for a successor to quantum mechanics – whether to solve these interpretative problems or to reconcile the quantum with general relativity's treatment of gravitation, or both (such as 't Hooft, see Chap. 8) – may scrutinize the details of this empirical success for clues about how present-day quantum mechanics might be an effective, i.e., phenomenological, theory. As 't Hooft wittily puts it: we can ask, not 'Quo vadis, quantum mechanics?', but rather 'Unde venis?' – 'Where do you come from?'

This paper will likewise ask 'Unde venis, quantum mechanics?': though I humbly admit that I will interpret this question in a retrospective and expository sense, rather than in 't Hooft's wonderfully forward-looking and creative sense. To be specific, I propose to discuss Hamilton–Jacobi theory as a classical root of quantum mechanics.

One part of this story is well known to physicists and philosophers and historians of physics. Namely, Hamilton–Jacobi theory as a method of integrating Hamilton's equations (using Jacobi's theorem, action–angle variables, etc.), and the use made of this integration theory in nineteenth century celestial mechanics, and thereby in the old quantum theory.

There is however another part of this story that seems much less known by this community, viz., Hamilton–Jacobi theory understood from the perspective of the calculus of variations (as worked out by such masters as Hilbert and Carathéodory), and how this understanding motivates de Broglie's and Schrödinger's proposal to extend Hamilton's optico-mechanical analogy, thus creating quantum mechanics (as wave mechanics). So I propose to present

this part of the story: or rather, since this part could fill a book – selected pieces of it! [My (2003, 2003a) discuss some other, philosophical, aspects.] At the end of the paper, I shall also briefly return to 'Quo vadis?', i.e., to a current interest in the foundations of quantum theory, viz., the pilot-wave theory – on which Hamilton–Jacobi theory casts some light. But I begin, in the next section, with a more detailed prospectus.

13.1.1 Introducing Hamilton–Jacobi Theory

Hamilton–Jacobi theory is a general theory, rich in analytic and geometric ideas, that unifies three apparently disparate topics: systems of first order ordinary differential equations, first order partial differential equations, and the calculus of variations. Roughly speaking, Hamilton–Jacobi theory shows that the following problems are equivalent:

- (ODE): solving a canonical system of first order ordinary differential equations ($2n$ equations for $2n$ functions of a parameter t in which all variables' first derivatives are given by partial derivatives of one and the same function); e.g., Hamilton's equations in Hamiltonian mechanics.
- (PDE): solving a first order partial differential equation in which the unknown function does not occur explicitly, e.g., the Hamilton–Jacobi equation in mechanics.
- (CV): solving the 'basic' calculus of variations problem of finding n functions $q_1, \ldots, q_n$ of a parameter t that make stationary a line integral of the form $\int L(q_i, \dot{q}_i, t)\mathrm{d}t$, where the dot denotes differentiation with respect to t, e.g., Hamilton's principle in Lagrangian mechanics, or Fermat's principle in geometric optics.

A bit more precisely: elementary Lagrangian and Hamiltonian mechanics show (ODE) and (CV) to be equivalent for the case of fixed endpoints. Hamilton–Jacobi theory extends this equivalence by considering, not a single solution of the canonical equations (a single line integral) but a whole field of solutions, i.e., line integrals along all the curves of a space-filling congruence (so that the endpoints lie on hypersurfaces transverse to the congruence). The initial conditions of a problem then become the specification of a function's values on such a hypersurface, instead of an initial configuration and momentum (or an initial and final configuration): hence the occurrence of partial differential equations.

The main aim of this paper is to explain (in an elementary way) these equivalences and some related results. This explanation will later (Sects. 13.7 and 13.8) provide us with a perspective on the optico-mechanical analogy and quantum mechanics (specifically, wave mechanics). But there is also a pedagogic rationale for presenting these results. Most physicists learn Hamilton–Jacobi theory only as part of analytical mechanics; and almost all the mechanics textbooks present, in addition to the equivalence of (ODE) and (CV) for fixed endpoints, only the use of Hamilton–Jacobi theory as a method of

integrating Hamilton's equations – indeed rendering the integration trivial. The central result here is Jacobi's theorem: that given a complete integral of the Hamilton–Jacobi equation (typically found by separation of variables), one can obtain solutions of Hamilton's equations just by differentiation. This is a remarkable result, which lies at the centre of a beautiful geometric theory of the integration of first order partial differential equations, a theory which reduces the integration problem to that of integrating a suitable system of ordinary differential equations (the characteristic equations). But almost all the mechanics textbooks present Jacobi's theorem using just canonical transformation theory. As a result, they do not describe this general integration theory – and more generally, they do not show the role of geometric ideas, nor of the calculus of variations with variable endpoints.

This textbook tradition is of course understandable. Textbooks must emphasise problem-solving; and the use of a complete integral of the Hamilton–Jacobi equation to solve Hamilton's equations is crucially important, for several reasons. As to problem-solving, it is 'the most powerful method known for exact integration, and many problems which were solved by Jacobi cannot be solved by other means' (Arnold 1989, p. 261). Besides, it is conceptually important. It leads on to action–angle variables, which are central both to classical mechanics (e.g., in the Liouville–Arnold theorem, and in perturbation theory) and the old quantum theory.

But though understandable, this tradition is also regrettable. For the result is that most physicists understand well only the equivalence of (ODE) and (CV) for fixed endpoints, and a part of the equivalence of (PDE) and (ODE) – the part expressed by Jacobi's theorem. Besides, they understand these matters only in the context of mechanics. This is a pity, for two reasons.

First, it is worth stressing that all these equivalences and other related results are purely mathematical and so entirely general. Second, the equivalences and results that get omitted from most mechanics textbooks are at least as rich as those included; in particular, in their use of geometric ideas and one might add, in their use of optical ideas. Indeed, Hamilton developed his work in mechanics in deliberate analogy with his previous work in optics.[1] And as we shall see, both Fermat's principle (roughly, that a light ray travels the path that takes least time) and Huygens's principle (roughly, that given a wave front, a later wave front is the envelope of spherical waves spreading from the points of the given wave front) stand at the centre of Hamilton–Jacobi theory. They involve each of the above mathematical problems, in optical guise, viz., the description of light in terms of rays [exemplifying (ODE)], in terms of wavefronts [exemplifying (PDE)], and by means of variational principles [exemplifying (CV)].

[1] For a glimpse of the history, which I will not discuss, see for example: for mechanics, Dugas (1988), Whittaker (1959); for optics, Whittaker (1952), Buchwald (1989); and for mathematics: Kline (1970, Chap. 30).

Accordingly, I propose to expound some of these equivalences and connections, as mathematics (Sects. 13.2–13.6). Then I will illustrate them with geometric optics and the optico-mechanical analogy (Sects. 13.7 and 13.8).

To be both brief and elementary, this exposition must be very selective. In particular, I will say nothing about:

(i) weak solutions,
(ii) the use of phase space,
(iii) issues about the global existence of solutions, including focussing and caustics.[2]

Another omitted topic lies closer to our concerns. I will not present the theory surrounding Jacobi's theorem, i.e., Hamilton–Jacobi theory as an integration theory for first order partial differential equations. For though I have complained that this is absent from the mechanics books, it is in some books on mathematical methods.[3]

Instead, I will adopt an approach that emphasises the calculus of variations. The main ideas here seem to be due to Carathéodory and Hilbert. Here again, I must be selective. I will simply pick out within this approach, one line of thought, found for example in the first half of Rund (1966). (Rund proves some results which I will only state, and he cites the original papers.) Though selective, this exposition will give a good sense of the triangle of equivalences between (ODE), (PDE) and (CV). Indeed, we will get such a sense already by the end of Sect. 13.3. Sections 13.4 to 13.6 will add to this a discussion of three topics, each leading to the next. They are, respectively, Hilbert's independent integral, treating the integration variable of the variational problem on the same footing as the other coordinates, and integration theory.

Thereafter, Sects. 13.7 et seq. return us to physics. Section 13.7 discusses geometric optics, and Sect. 13.8 the optico-mechanical analogy and wave mechanics. Section 13.8 also leads us back to the foundations of quantum

[2] A few pedagogic references: for (i) Logan (1994, Chap. 3), Stakgold (1967); for (ii), Arnold (1989, Chaps. 8 and 9), Littlejohn (1992), Taylor (1996, Sect. 1.15); for (iii), Arnold (1989, Appendices 11 and 16), Benton (1977), Taylor (1996, Sect. 6.7). Of these topics, (ii) and (iii) are closest to this paper's interests in geometry, and in the transition between classical and quantum mechanics. For (ii), i.e., Hamilton–Jacobi theory in phase space, beautifully illustrates symplectic geometry; and (ii) and (iii) are crucial in both quantization theory and semiclassical mechanics.

[3] Especially Courant and Hilbert (1962, Sects. II.1–8); see also, e.g., Webster (1950, Chap. 2) and John (1971, Chap. 1). In order to be elementary, I will also avoid all use of modern differential geometry, including even the distinction between contravariant and covariant indices. Though modern geometry has transformed our understanding of differential equations and the calculus of variations (and the sciences of mechanics and optics), I shall only need the intuitive geometry familiar from multivariable calculus.

mechanics, which I take up briefly in (the last) Sect. 13.9. Here I will call attention to the role of Hamilton–Jacobi theory in the pilot-wave theory of de Broglie and Bohm, and more specifically, advertise Holland's recent work (2001, 2001a), which provides a Hamiltonian formulation of the pilot-wave theory.

13.2 From the Calculus of Variations to the Hamilton–Jacobi Equation

13.2.1 The Calculus of Variations Reviewed

We begin by briefly reviewing the simplest problem of the calculus of variations, with which we will be concerned throughout the paper. This is the variational problem (in a notation suggestive of mechanics)

$$\delta I := \delta I[q_i] = \delta \int_{t_0}^{t_1} L(q_i, \dot{q}_i, t)\, \mathrm{d}t = 0\ , \quad i = 1, \ldots, n\ , \tag{13.1}$$

where [ ] indicates that I is a functional, the dot denotes differentiation with respect to t, and L is to be a C^2 (twice continuously differentiable) function in all $2n + 1$ arguments. L is the *Lagrangian* or *fundamental function*, and $\int L\, \mathrm{d}t$ is the *fundamental integral*. We will discuss this only locally, i.e., we will consider a fixed simply-connected region G of an $(n+1)$-dimensional real space $\mathbb{R}^{n+1}$, on which there are coordinates $(q_1, \ldots, q_n, t) =: (q_i, t) =: (q, t)$.

The singling out of a coordinate t (called the *parameter* of the problem), to give a parametric representation of curves $q(t) := q_i(t)$, is partly a matter of notational clarity. But it is of course suggestive of the application to mechanics, where t is time, q represents the system's configuration and (q_i, t)-space is often called extended configuration space or event space. Besides, the singling out of t reflects the fact that, though it is usual to assume that L (and so the fundamental integral) is invariant under arbitrary transformations (with non-vanishing Jacobian) of the q_i, we do not require the fundamental integral to be independent of the choice of t. Indeed we shall see (at the end of this section and in Sect. 13.5) that allowing this dependence is necessary for making Legendre transformations.[4]

A necessary condition for I to be stationary at the C^2 curve $q(t) := q_i(t)$, i.e., for $\delta I = 0$ in comparison with other C^2 curves that

- share with $q(t)$ the endpoints $q(t_0)$, $q(t_1)$,

[4] Of course, the calculus of variations, and Hamilton–Jacobi theory, can be developed on the assumption that the fundamental integral is to be parameter-independent – if it could not be, so much the worse for relativistic theories! But the details, in particular of how to set up a canonical formalism, are different from what follows. For these details, see for example Rund (1966, Chap. 3).

- are close to $q(t)$ in both value and derivative throughout $t_0 < t < t_1$,

is that $q(t)$ satisfies for $t_0 < t < t_1$ the n second-order Euler–Lagrange (also known as Euler, or as Lagrange!) equations

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{\dot{q}_i} - L_{q_i} = 0 \,, \quad i = 1, \ldots, n \,. \tag{13.2}$$

A curve satisfying these equations is called an *extremal*.

We will not need to linger on the usual derivation of these equations: we will later see them derived without using a single fixed pair of endpoints. Nor need we linger on several related matters taken up in the calculus of variations, such as: the distinction between stationarity and extrema (i.e., maxima or minima), in particular the conditions for a curve to be an extremum not just a stationary point (e.g., conditions concerning the second variation of the fundamental integral, or Weierstrass' excess function); the distinction between weak and strong stationary points and extrema; and the use of weaker assumptions about the smoothness of the solution and comparison curves.

But it is important to consider the canonical form of our variational problem. In physics, the most frequent example of this is the expression of Hamilton's principle within Hamiltonian mechanics, i.e., Hamilton's principle with the integrand a function of both qs and ps, which are to be varied independently. But the correspondence between the Lagrangian form of the variational problem (above) and the canonical form is general (purely mathematical).

Thus, under certain conditions the variational problem (13.1) has an equivalent form, whose Euler–Lagrange equations are $2n$ first order equations. To this end, we introduce 'momenta'

$$p_i := L_{\dot{q}_i} \,, \tag{13.3}$$

and (recalling that L is C^2) we assume that the Hessian with respect to the $\dot{q}$s does not vanish in the domain G considered, i.e., the determinant

$$\left| L_{\dot{q}_i \dot{q}_j} \right| \neq 0 \,, \tag{13.4}$$

so that (13.3) can be solved for the $\dot{q}_i$ as functions of q_i, p_i, t.

Then the equations

$$p_i = L_{\dot{q}_i} \,, \quad \dot{q}_i = H_{p_i} \,, \quad L(q_i, \dot{q}_i, t) + H(q_i, p_i, t) = \sum_i \dot{q}_i p_i \,, \tag{13.5}$$

represent a *Legendre transformation* and its inverse, where, in the third equation, $\dot{q}_i$ are understood as functions of q_i, p_i, t according to the inversion of (13.3). The function $H(q_i, p_i, t)$ is called the *Legendre* (or *Hamiltonian*) function of the variational problem, and the qs and ps are called *canonically*

conjugate. It follows that H is C^2 in all its arguments, $H_t = -L_t$, and $|L_{\dot{q}_i \dot{q}_j}| = |H_{p_i p_j}|^{-1}$. Besides, any $H(q_i, p_i, t)$ that is C^2 in all its arguments and has a non-vanishing Hessian with respect to the ps, $|H_{p_i p_j}| \neq 0$, is the Legendre function of a C^2 Lagrangian L given in terms of H by (13.5).

Applying this Legendre transformation, the Euler–Lagrange equations (13.2) go over to the *canonical system*

$$\dot{q}_i = H_{p_i} \ , \quad \dot{p}_i = -H_{q_i} \ (= L_{q_i}) \ . \tag{13.6}$$

A curve satisfying these equations is also called an extremal. These are the Euler–Lagrange equations of a variational problem equivalent to the original one, in which both qs and ps are varied independently, namely the problem

$$\delta \int \left[\sum_i \dot{q}_i p_i - H(q_i, p_i, t) \right] \mathrm{d}t = 0 \ . \tag{13.7}$$

For more details about (13.3)–(13.7), see for example Arnold (1989, Sects. 3.14, 9.45.C), Courant and Hilbert (1953, Sect. IV.9.3; 1962, Sect. I.6) and Lanczos (1986, Sects. VI.1–4).

The requirement of a non-vanishing Hessian, (13.4), or equivalently $|H_{p_i p_j}| \neq 0$, is a crucial assumption. Note in particular these two consequences:

- The Hamiltonian cannot vanish identically. This follows because, if we differentiate $H = \sum \dot{q}_i p_i - L = 0$ with respect to $\dot{q}_i$, we get $\sum_i L_{\dot{q}_i \dot{q}_j} \dot{q}_i = 0$, and this contradicts (13.4).
- L cannot be homogeneous of the first degree in the $\dot{q}_i$. That is, we cannot have $L(q_i, \lambda \dot{q}_i, t) = \lambda L(q_i, \dot{q}_i, t)$. We shall see in Sect. 13.5 that this means the fundamental integral cannot be parameter-independent.

13.2.2 Hypersurfaces and Congruences

We consider a family of hypersurfaces in our region G of $\mathbb{R}^{n+1}$,

$$S(q_i, t) = \sigma \ , \tag{13.8}$$

with $\sigma \in \mathbb{R}$ the parameter labelling the family, and S a C^2 function (in all $n+1$ arguments). We assume this family covers the region G *simply*, in the sense that through each point of G there passes a unique hypersurface in the family.

Let C be a curve

$$q_i = q_i(t) \tag{13.9}$$

of class C^2, that lies in G and intersects each hypersurface in the family (13.8) just once, but is nowhere tangent to a hypersurface. Then σ is a function of t along C, with

$$\Delta := \frac{\mathrm{d}\sigma}{\mathrm{d}t} = \sum_i \frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial t} \ . \tag{13.10}$$

By construction $\Delta \neq 0$. We will assume that the Lagrangian L does not vanish along C. By a suitable labelling of the family of surfaces, we can secure

$$\Delta > 0 \ \text{ or } \ < 0 \ \text{ according as } \ L > 0 \ \text{ or } < 0 \tag{13.11}$$

for the line element $(q_i, \dot{q}_i, t)$ of C. Then a tangential displacement along C from $P := (q_i, t)$ to $Q := (q_i + \mathrm{d}q_i, t + \mathrm{d}t)$, i.e., a displacement with components $(\mathrm{d}q_i, \mathrm{d}t) = (\dot{q}_i, 1)\mathrm{d}t$, induces an increment $\mathrm{d}\sigma$ in σ, and an increment $\mathrm{d}I = L(q_i, \dot{q}_i, t)\,\mathrm{d}t$ in $I = \int L\,\mathrm{d}t$.

To connect this family of hypersurfaces with the calculus of variations, we now seek values of $\dot{q}_i$ at P such that the direction at P of the curve C, $(\dot{q}_i, 1)\,\mathrm{d}t$, makes $\mathrm{d}I/\mathrm{d}\sigma$ a minimum with $\mathrm{d}\sigma$ fixed. A necessary condition is that

$$\frac{\partial}{\partial \dot{q}_i}\left(\frac{\mathrm{d}I}{\mathrm{d}\sigma}\right) = 0 \ , \quad i = 1, \ldots, n \ . \tag{13.12}$$

But $\mathrm{d}I/\mathrm{d}\sigma = L/\Delta$ and $\Delta \neq 0$, so that (13.12) reads

$$\frac{\partial L}{\partial \dot{q}_i} - \frac{L}{\Delta}\frac{\partial \Delta}{\partial \dot{q}_i} = 0 \ . \tag{13.13}$$

That is, using $\partial\Delta/\partial\dot{q}_i = \partial S/\partial q_i$ from (13.10),

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{L}{\Delta}\frac{\partial S}{\partial q_i} \ . \tag{13.14}$$

A curve C, or its tangent vector $(\dot{q}_i, 1)$, that satisfies (13.14), is said to be in the direction of the *geodesic gradient determined by the family of surfaces* (13.8).

As it stands, this condition (13.14) can at best yield minima of $\mathrm{d}I/\mathrm{d}\sigma$, while we are interested in minima of $\mathrm{d}I/\mathrm{d}t$. But there is a further condition on the family of surfaces (13.8) that implies that curves obeying (13.14) are solutions of the variational problem; or rather, to be precise, extremals.

This condition has two equivalent forms, the first geometric in spirit, the second analytic. They are:

(a) The quantity L/Δ is constant on each surface, i.e., there is some real function ϕ such that

$$\frac{L}{\Delta} = \phi(\sigma) \ , \tag{13.15}$$

where we are to take the directional arguments in L to refer to the geodesic gradient.

(b) S solves the Hamilton–Jacobi equation.

It is straightforward to show that (a) implies that we can reparametrize the family of surfaces in such a way that $L = \Delta$ throughout the region G. That is to say, given (a), the family can be reparametrized so that function ϕ is the constant function 1: $\phi(\sigma) = 1$. Proof: Any monotonic function ψ gives a reparametrization of the family, $\psi(S) = \psi(\sigma)$, with $\bar{\Delta}$ defined on analogy with Δ by $\bar{\Delta} := \mathrm{d}\psi(\sigma)/\mathrm{d}t = \psi'(\sigma)\Delta$. Choosing $\psi(\sigma) := \int_{\sigma_0}^{\sigma} \phi(s)\,\mathrm{d}s$ (σ_0 some constant) yields $\psi'(\sigma) = \phi(\sigma)$ so that

$$\frac{L}{\bar{\Delta}} \equiv \frac{L}{\psi'(\sigma)\Delta} \equiv \frac{\phi(\sigma)}{\psi'(\sigma)} = 1 \,.$$

So to show (a) and (b) equivalent, we will show that:

(i) given (a) in this special form, i.e., given $L = \Delta$, S solves the Hamilton–Jacobi equation,

and conversely,

(ii) S solving the Hamilton–Jacobi equation implies that $L = \Delta$.

But it will be clearest, before proving this equivalence, to present two consequences of $L = \Delta$, and introduce some terminology.

Firstly, $L = \Delta$ implies that the geodesic gradient (13.14) is now given by

$$\frac{\partial L}{\partial \dot{q}_i} = p_i = \frac{\partial S}{\partial q_i} \,, \tag{13.16}$$

where the first equation uses (13.3). Recall now our assumption that the determinant $|L_{\dot{q}_i \dot{q}_j}| \neq 0$, so that (13.3) can be solved in G for the $\dot{q}_i$ as functions of q_i, p_i, t, viz., $\dot{q}_i = q_i(q_i, p_i, t)$. This now reads as

$$\dot{q}_i = \dot{q}_i(q_i, \partial S/\partial q_i, t) \,, \tag{13.17}$$

where the right-hand side is a function of (q_i, t) alone (since S is) and has continuous first order derivatives. Then the elementary existence theorem for solutions of first order ordinary differential equations implies that (13.17) defines an n-parameter family of curves in the region G, such that each point in G has a unique curve pass through it, and each curve is a solution of (13.17) in the sense that the components of its tangent vectors obey (13.17). This family of curves is called the *congruence K belonging to the family of surfaces* (13.8).

Secondly, $L = \Delta$ implies that the increment $\mathrm{d}I$ in the fundamental integral $I = \int L\,\mathrm{d}t$, in passing from a point P_1 on the surface $S(q_i, t) = \sigma_1$, to an adjacent surface $S = \sigma_1 + \mathrm{d}\sigma$, along a curve of the congruence belonging to the family, obeys

$$\mathrm{d}I = \Delta \mathrm{d}t = \mathrm{d}\sigma \,. \tag{13.18}$$

Integrating this result along members of the congruence, we find that the integral along a curve of the congruence, from any point P_1 on the surface $S(q_i, t) = \sigma_1$ to that point P_2 on the surface $S(q_i, t) = \sigma_2$ that lies on the same curve of the congruence, is the same for whatever point P_1 we choose. That is,

$$\int_{P_1}^{P_2} L \, \mathrm{d}t = \sigma_2 - \sigma_1 \, . \tag{13.19}$$

Clearly, the converse also holds: if the fundamental integral taken along curves of the congruence has the same value for two hypersurfaces, however we choose the endpoints P_1, P_2 lying in the hypersurfaces, then $L = \Delta$. So a family of surfaces satisfying the condition that $L = \Delta$ is called *geodesically equidistant* with respect to the Lagrangian L. [Courant and Hilbert (1962, Sect. II.9.2) say 'geodetic', not 'geodesic', which has the advantage of avoiding possibly confusing connotations of metric and/or connection with the word 'geodesic'.]

Carathéodory called a family of geodesically equidistant hypersurfaces, together with the congruence belonging to it, the *complete figure* (of the variational problem). As we shall see, the name is apt, since the complete figure is central to Hamilton–Jacobi theory. Also, the congruence is called *transversal* to the surfaces of the family. The analytical expression of transversality is that for a displacement $(\delta q_i, \delta t)$ tangential to a hypersurface in the family, $\delta S = 0$. That is,

$$\frac{\partial S}{\partial q_i} \delta q_i + \frac{\partial S}{\partial t} \delta t = 0 \, . \tag{13.20}$$

We turn to showing (i) above, viz., that $L = \Delta$ implies that S solves the Hamilton–Jacobi equation.

Proof. Equation (13.10) yields

$$L(q_i, \dot{q}_i, t) = \Delta := \frac{\mathrm{d}\sigma}{\mathrm{d}t} = \sum_i \frac{\partial S}{\partial q_i} \dot{q}_i + \frac{\partial S}{\partial t} \, , \tag{13.21}$$

where $\dot{q}_i$ refers to the direction of the geodesic gradient, (13.17), i.e., $\dot{q}_i = \dot{q}_i(q_i, \partial S/\partial q_i, t)$. This yields

$$-\frac{\partial S}{\partial t} = -L\big(q_i, \dot{q}_i(q_i, \partial S/\partial q_i, t), t\big) + \sum_i \frac{\partial S}{\partial q_i} \dot{q}_i(q_i, \partial S/\partial q_i, t) \, . \tag{13.22}$$

However, (13.5) implies that the right-hand side is the Hamiltonian function, but with p_i replaced by $\partial S/\partial q_i$ in accordance with (13.16). Thus we have

$$\frac{\partial S}{\partial t} + H(q_i, \partial S/\partial q_i, t) = 0 \, , \tag{13.23}$$

which is the *Hamilton–Jacobi equation*.

This equation is also a sufficient condition for a family of surfaces being geodesically equidistant. That is, (ii) above: S being a C^2 solution in G of the Hamilton–Jacobi equation implies that $L = \Delta$, i.e., that the hypersurfaces of constant S are geodesically equidistant.

Proof. Given such a solution $S(q_i, t)$, let us define an assignment to each point of G (sometimes called a *field*) by

$$p_i \equiv p_i(q_i, t) := \frac{\partial S}{\partial q_i} \; . \tag{13.24}$$

By (13.4), this determines a field $\dot{q}_i$ as in (13.17), and hence a congruence. Then for the given solution S, a given member C of the congruence, and two given parameter values σ_1, σ_2, we form the fundamental integral along C between the points P_1 and P_2 where C intersects the hypersurfaces corresponding to the parameter values σ_1, σ_2. Using the Legendre transformation (13.5) and the fact that S solves the Hamilton–Jacobi equation (13.23), we obtain

$$\begin{aligned} \int_{P_1}^{P_2} L \, \mathrm{d}t &= -\int_{P_1}^{P_2} \left[H(q_i, \partial S/\partial q_i, t) - \sum_i p_i \dot{q}_i \right] \mathrm{d}t \\ &= \int_{P_1}^{P_2} \left(\frac{\partial S}{\partial t} \mathrm{d}t + \sum_i \frac{\partial S}{\partial q_i} \mathrm{d}q_i \right) \\ &= \int_{P_1}^{P_2} \mathrm{d}S = \sigma_2 - \sigma_1 \; . \end{aligned} \tag{13.25}$$

To sum up, a family of hypersurfaces $S = \sigma$ is geodesically equidistant with respect to the Lagrangian L iff S is a solution of the Hamilton–Jacobi equation whose Hamiltonian H corresponds by the Legendre transformation to L. And if this holds, the transversality condition[5] (13.20) can be written [using (13.23) and (13.24)] as

$$p_i \delta q_i - H(q_i, p_i, t) \delta t = 0 \; . \tag{13.26}$$

13.3 Canonical and Euler–Lagrange Equations. Fields of Extremals

We now study the properties of a congruence K belonging to a family of geodesically equidistant surfaces. We first show that any curve of such a con-

[5] Transversality can also be defined, without any use of a family of hypersurfaces, or even a function S, in terms of the fundamental integral being stationary as an endpoint of the integral varies on a given single surface [see, e.g., Courant and Hilbert (1953, Sect. IV.5.2)].

gruence obeys the canonical and Euler–Lagrange equations. Then we develop the following ideas: a field q_i, p_i in the region G; and a field belonging to a family of (not necessarily geodesically equidistant) hypersurfaces. Finally we characterize those fields belonging to geodesically equidistant hypersurfaces.

13.3.1 Canonical and Euler–Lagrange Equations

The family (13.8) defines an assignment of $p_i := \partial S/\partial q_i$ to each point of a member C of the congruence K. If we differentiate the definition of p, i.e., (13.24) with respect to t along C, then differentiate the Hamilton–Jacobi equation (13.23), and use the fact [from (13.6)] that $\dot{q}_i = \partial H/\partial p_i$, we can eliminate the second derivatives of S that arise in the differentiations, and get

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \,. \tag{13.27}$$

To this, we can adjoin $\dot{q}_i = \partial H/\partial p_i$, so as to get $2n$ first order ordinary differential equations obeyed by members of K, viz.,

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \,, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i} \,. \tag{13.28}$$

Note that according to this deduction, these two groups of equations have different statuses, despite their symmetric appearance. $\dot{p}_i = -\partial H/\partial q_i$ depends on K belonging to a family of geodesically equidistant surfaces (i.e., on the Hamilton–Jacobi equation), whereas $\dot{q}_i = \partial H/\partial p_i$ are identities derived from the theory of the Legendre transformation [see (13.6)]. But this difference is not peculiar to the use of hypersurfaces in our deduction. The same difference occurs in derivations of these equations in the calculus of variations with fixed endpoints: in the most familiar case, in Lagrangian mechanics, i.e., without use of the canonical integral [see, e.g., Lanczos (1986, pp. 166–7)].

From the canonical equations we can deduce the (Lagrangian form of the) Euler–Lagrange equations. We substitute $p_i = \partial L/\partial \dot{q}_i$ in the left-hand side, and $\partial H/\partial q_i = -\partial L/\partial q_i$ in the right-hand side, of the first of (13.28), to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \,. \tag{13.29}$$

13.3.2 Fields

To discuss fields, we need first to consider parametric representations of an arbitrary smooth congruence of curves covering our region G simply. That is, we consider a congruence represented by n equations giving q_i as C^2 functions of n parameters and t,

$$q_i = q_i(u_\alpha, t)\ , \qquad i = 1, \ldots, n\ , \tag{13.30}$$

where each set of n $u_\alpha = (u_1, \ldots, u_n)$ labels a unique curve in the congruence. Thus there is a one-to-one correspondence $(q_i, t) \leftrightarrow (u_\alpha, t)$ in appropriate domains of the variables, with non-vanishing Jacobian

$$\left| \frac{\partial q_i}{\partial u_\alpha} \right| \neq 0\ . \tag{13.31}$$

Such a congruence determines tangent vectors $(\dot{q}_i, 1)$ at each (q_i, t); and thereby also values of the Lagrangian $L\big(q_i(u_\alpha, t), \dot{q}_i(u_\alpha, t), t\big)$ and of the momentum

$$p_i = p_i(u_\alpha, t) = \frac{\partial L}{\partial \dot{q}_i}\ . \tag{13.32}$$

Conversely, a set of $2n$ C^2 functions q_i, p_i of (u_α, t) as in (13.30) and (13.32), with the qs and ps related by $p_i = \partial L/\partial \dot{q}_i$, determines a set of tangent vectors, and so a congruence. Such a set of $2n$ functions is called a field; and if all the curves of the congruence are extremals (i.e., solutions of the Euler–Lagrange equations), it is called a *field of extremals*.

We say a field *belongs to a* (not necessarily geodesically equidistant) *family of hypersurfaces* given by (13.8) iff throughout the region G (13.16) and (13.30) are together satisfied, i.e., iff we have

$$p_i = \frac{\partial}{\partial q_i} S(q_i, t) = \frac{\partial}{\partial q_i} S\big(q_i(u_\alpha, t), t\big)\ . \tag{13.33}$$

One can show that a field belongs to a family of hypersurfaces iff for all indices $\alpha, \beta = 1, \ldots, n$, the Lagrange brackets of the parameters of the field, i.e.,

$$[u_\alpha, u_\beta] := \sum_i \left(\frac{\partial q_i}{\partial u_\alpha} \frac{\partial p_i}{\partial u_\beta} - \frac{\partial q_i}{\partial u_\beta} \frac{\partial p_i}{\partial u_\alpha} \right)\ , \tag{13.34}$$

vanish identically.[6]

We say that a field $q_i = q_i(u_\alpha, t)$, $p_i = p_i(u_\alpha, t)$ is *canonical* if the q_i, p_i satisfy (13.28). Now we will show that if a canonical field belongs to a family of hypersurfaces (13.8), then the members of the family are geodesically equidistant.

Proof. Differentiating (13.33) with respect to t along a member of the congruence, and substituting on the left-hand side from the first of (13.28), we get

[6] See Rund (1966, pp. 28–30). Warning: the role of Lagrange brackets in this theory is sometimes omitted even in excellent accounts, e.g., Courant and Hilbert (1962, Sect. II.9.4).

$$-\frac{\partial H}{\partial q_i} = \frac{\partial^2 S}{\partial q_i \partial q_j}\dot{q}_j + \frac{\partial^2 S}{\partial q_i \partial t} \,. \tag{13.35}$$

By the second of (13.28), this is

$$\frac{\partial^2 S}{\partial q_i \partial t} + \frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial p_j}\frac{\partial^2 S}{\partial q_i \partial q_j} = 0 \,, \tag{13.36}$$

which is

$$\frac{\partial}{\partial q_i}\left[\frac{\partial S}{\partial t} + H(q_j, \partial S/\partial q_j, t)\right] = 0 \,. \tag{13.37}$$

This can be integrated immediately to give

$$\frac{\partial S}{\partial t} + H(q_j, \partial S/\partial q_j, t) = f(t) \,, \tag{13.38}$$

with f an arbitrary function of t only. Now we argue (in the usual way, for the calculus of variations) that this function can be absorbed in H. For suppose the given Lagrangian were replaced by $\tilde{L} = L + f(t)$. The path-independence of the integral $\int f(t)\,\mathrm{d}t$ implies that L and $\tilde{L}$ give equivalent variational problems, i.e., the same curves give stationary values for both $\int L\,\mathrm{d}t$ and $\int \tilde{L}\,\mathrm{d}t$. Besides, the definition (13.3) of p_i and the canonical equations (13.6) are unaffected, the only change in our formalism being that H is replaced by $\tilde{H} = H - f(t)$. So assuming that L is replaced by $\tilde{L}$ means that (13.38) reduces to the Hamilton–Jacobi equation (13.23). The result now follows from result (ii) at the end of Sect. 13.2.2.

This result is a kind of converse of our deduction of (13.28). We can sum up this situation by saying that the canonical equations characterize any field belonging to a family of geodesically equidistant hypersurfaces.

Finally, we should note an alternative to our order of exposition. We assumed at the outset a family of hypersurfaces and then discussed an associated congruence and field. But one can instead begin with a single arbitrary surface; then define the notion of an extremal being transverse to the surface (in terms of the fundamental integral being stationary as an endpoint varies on the surface – see footnote 5); then define a field of such transverse extremals; and finally define other surfaces, geodesically equidistant to the given one, as surfaces $S =$ constant, where $S(q_i, t)$ is defined to be the value of the fundamental integral taken along a transverse extremal from the given surface ($S = 0$) to the point (q_i, t). This alternative order of exposition is adopted by Courant and Hilbert (1962, Sect. II.9.2-5), and (more briefly) by Born and Wolf (1999, Appendix I.2-4). It has the mild advantage over ours of clearly displaying the choice of an arbitrary initial surface; which accords with the solution of a partial differential equation involving an arbitrary function just as the solution of an ordinary differential equation involves an arbitrary constant or constants. It will also come up again in Sects. 13.6 and 13.7.

13.4 Hilbert's Independent Integral

A canonical field belonging to a geodesically equidistant family of hypersurfaces defines a line integral which is independent of its path of integration. This integral, named after its discoverer Hilbert, is important not only in Hamilton–Jacobi theory, but also in aspects of the calculus of variations which we do not discuss, e.g., the study of conditions for the fundamental integral to take extreme values.

Suppose we are given a geodesically equidistant family of hypersurfaces covering region G simply. Consider two arbitrary points $P_1, P_2 \in G$ lying on hypersurfaces $S = \sigma_1$, $S = \sigma_2$, respectively; and consider an arbitrary C^1 curve $C : q_i = q_i(t)$ lying in G and joining P_1 and P_2. We will write the components of the tangent vector $(\mathrm{d}q_i/\mathrm{d}t, 1)$ of C as $(q_i', 1)$; for we continue to use the dot notation for differentiation along the geodesic gradient of the field belonging to S. Now consider the integral along C of $\mathrm{d}S$, so that the integral is trivially path-independent:

$$J := \int_{P_1}^{P_2} \mathrm{d}S(q_i, t) = \sigma_2 - \sigma_1 = \int_{P_1}^{P_2} \left(\frac{\partial S}{\partial q_i} q_i' + \frac{\partial S}{\partial t} \right) \mathrm{d}t \; . \tag{13.39}$$

We can apply $p_i = \partial S/\partial q_i$ and the Hamilton–Jacobi equation to the first and second terms of the integrand, respectively, to get a path-independent integral

$$J = \int_{P_1}^{P_2} \left[p_i q_i' - H(q_i, p_i, t) \right] \mathrm{d}t = \sigma_2 - \sigma_1 \; . \tag{13.40}$$

We can also Legendre transform to eliminate the p_i in favour of $\dot{q}_i$, getting

$$J = \int_{P_1}^{P_2} \left[L(q_i, \dot{q}_i, t) + \frac{\partial L}{\partial \dot{q}_i} \left(q_i' - \dot{q}_i \right) \right] \mathrm{d}t = \sigma_2 - \sigma_1 \; . \tag{13.41}$$

It is in this form that J is usually called the *Hilbert integral*.

A field

$$q_i = q_i(u_\alpha, t) \; , \qquad p_i = p_i(u_\alpha, t) \; , \tag{13.42}$$

assumed to belong to a family of hypersurfaces in the sense of (13.33), is called a *Mayer field* if substituting q_i, p_i in the integral in (13.40) yields an integral that is path-independent. So we have seen that a canonical field is a Mayer field. One can show that the converse holds, i.e., any Mayer field is canonical (Rund 1966, p. 33). So we have the result: a Mayer field is a canonical field belonging to a family of geodesically equidistant hypersurfaces. (It can also be shown that every extremal curve can be imbedded in a Mayer field.)

Combining this with the results of Sect. 13.3, we also have: the field (13.42) is a Mayer field iff the Lagrange brackets $[u_\alpha, u_\beta]$ vanish and the field obeys the canonical equations (13.28).

13.5 The Parameter as an Additional q-Coordinate

As we said at the start of Sect. 13.2.1, our theory has depended from the outset on the choice of t [see the fundamental integral (13.1)]. Indeed, we saw at the end of Sect. 13.2.1 that the non-vanishing Hessian (13.4) implies that L cannot be homogeneous of the first degree in the $\dot{q}_i$, i.e., we cannot have for all $\lambda \in \mathbb{R}$, $L(q_i, \lambda\dot{q}_i, t) = \lambda L(q_i, \dot{q}_i, t)$. And we shall shortly see that this implies that the fundamental integral cannot be parameter-independent.

But for some aspects of the theory, especially the discussion of Hamilton–Jacobi theory as an integration theory for first order partial differential equations in the next section, it is both possible and useful to treat t as a coordinate on a par with the q s. So in this section, we describe such a treatment and the gain in symmetry it secures.

To have some consistency with our previous notation, we first consider a Lagrangian $L(q_\alpha, \dot{q}_\alpha, t)$ with $n-1$ coordinates q_α, a parameter t and derivatives $\dot{q}_\alpha = \mathrm{d}q_\alpha/\mathrm{d}t$. So note that, in this section, Greek indices run from 1 to $n-1$. So the fundamental integral along a curve $C : q_\alpha = q_\alpha(t)$ in a suitable region G of $\mathbb{R}^n$ joining points P_1, P_2 with parameters t_1, t_2 is

$$I = \int_{t_1}^{t_2} L(q_\alpha, \dot{q}_\alpha, t)\,\mathrm{d}t\ . \tag{13.43}$$

Now we introduce a real C^1 function $\tau(t)$ which is such that $\mathrm{d}\tau/\mathrm{d}t > 0$ for all values of t under consideration, but is otherwise arbitrary. We write derivatives with respect to τ using dashes, so that

$$q'_\alpha = \dot{q}_\alpha \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)\ . \tag{13.44}$$

So with $\tau_1 := \tau(t_1)$, $\tau_2 := \tau(t_2)$, we can write (13.43) as

$$I = \int_{\tau_1}^{\tau_2} L\left(q_\alpha, q'_\alpha \frac{\mathrm{d}\tau}{\mathrm{d}t}, t\right) \frac{\mathrm{d}t}{\mathrm{d}\tau}\mathrm{d}\tau\ . \tag{13.45}$$

If we now write q_n for t, so that we can write the coordinates on $\mathbb{R}^n$ as

$$q_i = (q_\alpha, t) = (q_\alpha, q_n)\ , \quad i = 1, \ldots, n \quad \text{and} \quad \frac{\mathrm{d}t}{\mathrm{d}\tau} = q'_n \neq 0\ , \tag{13.46}$$

then we can write (13.45) as

$$I = \int_{\tau_1}^{\tau_2} L^*(q_i, q'_i)\,\mathrm{d}\tau\ , \tag{13.47}$$

where we have defined

$$L^*(q_i, q'_i) := L^*(q_\alpha, t, q'_\alpha, q'_n) := L\left(q_\alpha, \frac{q'_\alpha}{q'_n}, t\right)\ . \tag{13.48}$$

We stress that the values of the integrals (13.43) and (13.47) are equal. But the latter is by construction parameter-independent, since the choice of τ is essentially arbitrary. Also L^* is by construction positively homogeneous of the first degree in the $q_i' = (q_\alpha', q_n')$ – i.e., for all positive numbers λ, $L^*(q_i, \lambda q_i') = \lambda L(q_i, q_i')$ – irrespective of the form of the given Lagrangian L. In fact one can easily show that these two features are equivalent.

For the purposes of the next section, we now express the canonical equations of our variational problem, (13.43) or (13.47), in the new notation. But note that the total differentiation on the left-hand sides of the canonical equations will still be differentiation with respect to the original parameter t, hence indicated by a dot.

Writing the conjugate momenta of L^* as p^* for the moment, we have

$$p_\alpha^* = \frac{\partial L^*}{\partial q_\alpha'} = \frac{\partial L}{\partial \dot{q}_\alpha} \frac{1}{q_n'} q_n' = p_\alpha \ , \tag{13.49}$$

so that these are identical with the original conjugate momenta. We therefore drop the * in p_α^*. So the canonical equations for the indices $1, \ldots, n-1$ are given, with the original Hamiltonian (Legendre) function $H(q_\alpha, p_\alpha, t)$ as defined in (13.5), by

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \ , \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} \ . \tag{13.50}$$

On the other hand, for the new p_n, we have

$$p_n := \frac{\partial L^*}{\partial q_n'} = L - \sum_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} \frac{q_\alpha'}{q_n'} = L - \sum_\alpha p_\alpha \dot{q}_\alpha \ . \tag{13.51}$$

Comparing with the definition (13.5) of the Hamiltonian (Legendre) function, this is

$$p_n + H(q_\alpha, p_\alpha, t) = 0 \ . \tag{13.52}$$

So differentiating p_n with respect to the original parameter t along an extremal gives

$$\dot{p}_n := \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{\partial H}{\partial t} = -\frac{\partial H}{\partial q_n} \ , \tag{13.53}$$

which fits well with (13.50). (Here $-\mathrm{d}H/\mathrm{d}t = -\partial H/\partial t$ follows as usual from the canonical equations, i.e., from the Poisson bracket of H with itself vanishing identically.) But note that we also have $\dot{q}_n := \mathrm{d}t/\mathrm{d}t = 1 \neq \partial H/\partial p_n = -1$.

However, we can use the Hamilton–Jacobi equation to overcome this last 'wrinkle', i.e., to get a greater degree of symmetry. We can write the Hamilton–Jacobi equation of our variational problem (13.43) as

$$\Phi\left(q_i, \frac{\partial S}{\partial q_i}\right) = H\left(q_\alpha, \frac{\partial S}{\partial q_\alpha}, q_n\right) + \frac{\partial S}{\partial q_n} = 0 \ , \tag{13.54}$$

where Φ is defined as a function of $2n$ variables by

$$\Phi(q_i, p_i) := H(q_\alpha, p_\alpha, q_n) + p_n \ . \tag{13.55}$$

Now if the p_α in (13.55) refer to a field of extremals belonging to a solution $S(q_\alpha, q_n)$ of the Hamilton–Jacobi equation, so that $p_\alpha = \partial S/\partial q_\alpha$, then by (13.52) and (13.54), we also have $p_n = \partial S/\partial q_n$. Besides, (13.55) implies immediately

$$\frac{\partial \Phi}{\partial q_i} = \frac{\partial H}{\partial q_i} \ , \quad \frac{\partial \Phi}{\partial p_\alpha} = \frac{\partial H}{\partial p_\alpha} \ , \quad \frac{\partial \Phi}{\partial p_n} = 1 \quad (= \dot{q}_n \equiv \mathrm{d}t/\mathrm{d}t) \ . \tag{13.56}$$

It follows that we can write the canonical equations (13.50), together with the relations for q_n, p_n, in a completely symmetrical way in terms of Φ as

$$\dot{q}_i = \frac{\partial \Phi}{\partial p_i} \ , \quad \dot{p}_i = -\frac{\partial \Phi}{\partial q_i} \ , \tag{13.57}$$

where, note again, the dot denotes differentiation with respect to t.

13.6 Integrating First Order Partial Differential Equations

As mentioned in Sect. 13.1, we will not expound the usual approach (using Jacobi's theorem) to Hamilton–Jacobi theory as an integration theory for first order partial differential equations.[7] Instead, we will in this section briefly introduce another approach which exploits the results and concepts of the previous sections (for more details, see Rund, 1966, Sect. 2.8).

We will consider a partial differential equation of the form

$$\Phi\left(q_i, \frac{\partial S}{\partial q_i}\right) = 0 \ , \quad i = 1, \ldots, n \ , \quad \text{with} \quad \frac{\partial \Phi}{\partial p_i} \neq 0 \quad \text{for at least one } i \ , \tag{13.58}$$

and Φ of class C^2 in all $2n$ arguments. One of the i for which $\partial\Phi/\partial p_i \neq 0$ may be identified with t, but this is not necessary: as in the previous section, our discussion can treat all coordinates of $\mathbb{R}^n$ on an equal footing. We shall also assume that (as suggested by the Hamilton–Jacobi equation) the unknown function S does not occur explicitly in the equation; but this is not really a restriction, since one can show that the general case, i.e., an equation in which S occurs, can be reduced to the form of (13.58) by introducing an additional independent variable.

[7] For an exposition, see the references in footnote 4. As to the history, Whittaker (1959, pp. 264, 316) reports that this theory was developed by Pfaff and Cauchy in 1815–1819, using earlier results by Lagrange and Monge; i.e., well before Hamilton's and Jacobi's work!

So the initial value problem is to find a function $S(q_i)$ ($q_i \in G$) that satisfies (13.58) and that assumes prescribed values on a given $(n-1)$-dimensional C^2 surface, V say, in G. We will indicate how to construct such a function explicitly by using a congruence of 'canonical' curves which solve a canonical system of ordinary differential equations. (So we reduce the integration of the partial differential equation to the problem of integrating ordinary differential equations.) This canonical system of equations will be suggested by our previous discussion; and the strategy of the construction will be to adjust the congruence of curves from an initial rather arbitrary congruence, to one that provides a solution to (13.58).[8]

Thus our previous discussion (especially Sects. 13.3 and 13.5) suggests that we should consider the system of $2n$ ordinary differential equations, with a new parameter s:

$$\dot{q}_i := \frac{\mathrm{d}q_i}{\mathrm{d}s} = \frac{\partial \Phi(q_j, p_j)}{\partial p_i} \,, \qquad \dot{p}_i := \frac{\mathrm{d}p_i}{\mathrm{d}s} = -\frac{\partial \Phi(q_j, p_j)}{\partial q_i} \,. \tag{13.59}$$

These are called the *characteristic equations* of (13.58). A curve $q_i = q_i(s)$ of $\mathbb{R}^n$ that satisfies them is called a *characteristic curve* of (13.58). It will be an extremal of a problem in the calculus of variations if (13.58) is the Hamilton–Jacobi equation of such a problem. Our approach to integrating (13.58) applies to these characteristic equations theorems about the existence and uniqueness of solutions of ordinary differential equations, so as to secure the existence and uniqueness of solutions to (13.58).

Let us consider an $(n-1)$-parameter congruence of characteristic curves, with parameters $u_1, \ldots, u_{n-1}$, so that we write

$$q_i = q_i(s, u_\alpha) \,, \qquad p_i = p_i(s, u_\alpha) \,. \tag{13.60}$$

Since Φ is C^2, it follows from (13.59) that the functions (13.60) are C^2 in s. We will also assume that these functions are C^2 in the u_α, and that this congruence covers the region G simply, with

$$\frac{\partial(q_1, q_2, \ldots, q_n)}{\partial(s, u_1, \ldots, u_{n-1})} \neq 0 \,, \tag{13.61}$$

so that we can invert the first set of (13.60) for (s, u_α), getting

$$s = s(q_i) \,, \qquad u_\alpha = u_\alpha(q_i) \,. \tag{13.62}$$

We shall also write (in G)

$$\phi(s, u_\alpha) := \Phi\big(q_i(s, u_\beta), p_i(s, u_\beta)\big) \,. \tag{13.63}$$

One can now show:

[8] We remark at the outset that since – as in previous sections – we work in a 'configuration space', not its twice-dimensional 'phase space', there are many 'canonical congruences', rather than a unique one, so that this sort of adjustment is possible.

- ϕ of (13.63) is an integral of the characteristic equations (13.59), i.e., $\mathrm{d}\phi/\mathrm{d}s = 0$,
- the Lagrange brackets $[u_\alpha, s]$ and $[u_\alpha, u_\beta]$ are constant along any member of the congruence defined by (13.59).

We now make some assumptions about the relation of our characteristic congruence to the given surface V. We will assume that through each point of V there passes a unique member of the congruence, and that the congruence is nowhere tangent to V. Thus each point in V is assigned $n-1$ parameter values u_α and a value of s; so we can write s on V as a C^2 function of u_α, the parameters of the unique curve through the point. Let us write this as $s = \sigma(u_\alpha)$, so that the functions $a_i(u_\alpha)$ defined by

$$a_i(u_\alpha) := q_i\big(\sigma(u_\alpha), u_\alpha\big) \tag{13.64}$$

are also C^2. Finally we will suppose that we seek a solution of (13.58) which takes the values $c(u_\alpha)$ on V, c prescribed C^2 functions.

That completes the assumptions needed for the construction of a (local) solution of (13.58) (and the proof of its uniqueness). We end this section by briefly describing the first steps of the construction.

The theory of first order ordinary differential equations implies that the congruence of characteristic curves for (13.59) is determined if the values of q_i and p_i are prescribed on V. The initial values of q_i are of course to be given by the a_i of (13.64). But as to the initial values of the p_i, i.e., b_i defined by

$$b_i(u_\alpha) := p_i\big(\sigma(u_\alpha), u_\alpha\big)\ , \tag{13.65}$$

we have some choice. The strategy of the construction is, roughly speaking, to define a function S on G, in such a way that when we adjust the b_i so that $p_i = \partial S/\partial q_i$, S becomes a solution of (13.58) in G, possessing the required properties.

We now define a function $z = z(s, u_\alpha)$ on G in terms of V, the values $c(u_\alpha)$ prescribed on V and the given congruence; in effect, this z will be the desired S, once the b_i are suitably adjusted. For each point $P \in G$, with its n parameter values (s, u_α), the s-value of the intersection with V of the unique curve through P is given by $s = \sigma(u_\alpha)$. We define the value of z at P by

$$z(s, u_\alpha) := c(u_\alpha) + \int_{\tau=\sigma(u_\alpha)}^{s} \sum_i \left[p_i(\tau, u_\alpha) \frac{\partial \Phi\big(q_i(\tau, u_\alpha), p_i(\tau, u_\alpha)\big)}{\partial p_i} \right] \mathrm{d}\tau\ , \tag{13.66}$$

where the integration is to be taken along the curve through P, from its point of intersection with V, to P.

We will not go further into the construction of the desired S, except to make two remarks:

- Note that (13.66) implies in particular that $z(\sigma(u_\alpha), u_\alpha) = c(u_\alpha)$.

- Differentiating (13.66) with respect to s and using the first set of (13.59) yields

$$\dot{z} \equiv \frac{\partial z}{\partial s} = \sum_i p_i \frac{\partial \Phi}{\partial p_i} = \sum_i p_i \dot{q}_i \ . \tag{13.67}$$

This is analogous to the relation $\dot{S} = \sum_i p_i \dot{q}_i$ between a scalar function, such as a solution S of the Hamilton–Jacobi equation and the field q_i, p_i belonging to it, i.e., the field such that $p_i = \partial S / \partial q_i$ [see (13.33)]. Indeed, if we use (13.62) to define a function S on G by

$$S(q_i) := z\big(s(q_i), u_\alpha(q_i)\big) \ , \tag{13.68}$$

then one can show (again, we omit the details!) that:
- we can adjust the b_i so as to make $p_i = \partial S / \partial q_i$ hold,
- this adjustment makes S, as defined by (13.68) [and so (13.66)], a solution of (13.58) with the required properties.

13.7 The Characteristic Function and Geometric Optics

In this section, we follow in Hamilton's (1833, 1834!) footsteps. We introduce the Hamilton–Jacobi equation via the characteristic function (as do most mechanics textbooks), and then apply these ideas to geometric optics – so our discussion will (at last!) make contact with physics. The main point will be that the correspondence in our formalism between canonical extremals and geodesically equidistant hypersurfaces underpins the fact that both the corpuscular and wave conceptions of light can account for the phenomena, viz., reflection and refraction, described by geometric optics.[9]

We assume that our region $G \subset \mathbb{R}^{n+1}$ is sufficiently small that between any two points $P_1 = (q_{1i}, t_1)$, $P_2 = (q_{2i}, t_2)$ there is a unique extremal curve C. To avoid double subscripts, we will in this section sometimes suppress the i, writing $P_1 = (q_1, t_1)$, $P_2 = (q_2, t_2)$, etc. Then the value of the fundamental integral along C is a well-defined function of the coordinates of the endpoints, which we call the *characteristic function* and write as

$$S(q_1, t_1; q_2, t_2) = \int_{t_1}^{t_2} L \, \mathrm{d}t = \int_{t_1}^{t_2} \left(\sum_i p_i \dot{q}_i - H \right) \mathrm{d}t = \int \sum_i p_i \mathrm{d}q_i - H \mathrm{d}t \ , \tag{13.69}$$

[9] This is an example of what philosophers call under-determination of theory by data. The escape from this sort of quandary is of course the consideration of other phenomena: in this case, the nineteenth-century study of diffraction and interference, which led to the rise of wave optics – see the start of Sect. 13.8.

where the integral is understood as taken along the unique extremal C between the endpoints, and we have used (13.5).

Making arbitrary small displacements $(\delta q_1, \delta t_1)$, $(\delta q_2, \delta t_2)$ at P_1, P_2 respectively, and using the fact that the integral is taken along an extremal, we get for the variation in S

$$\begin{aligned}\delta S &:= S(q_1 + \delta q_1, t_1 + \delta t_1; q_2 + \delta q_2, t_2 + \delta t_2) - S(q_1, t_1; q_2, t_2) \\ &= \frac{\partial S}{\partial t_1}\delta t_1 + \frac{\partial S}{\partial t_2}\delta t_2 + \sum_i \frac{\partial S}{\partial q_{1i}}\delta q_{1i} + \sum_i \frac{\partial S}{\partial q_{2i}}\delta q_{2i} \\ &= \left[\sum_i p_i \delta q_i - H(q_j, p_j, t)\delta t\right]_{t_1}^{t_2} . \end{aligned} \tag{13.70}$$

Since the displacements are independent, we can identify each of the coefficients on the two sides of the last equation in (13.70), getting

$$\frac{\partial S}{\partial t_2} = -[H(q_i, p_i, t)]_{t=t_2} , \qquad \frac{\partial S}{\partial q_{2i}} = [p_i]_{t=t_2} , \tag{13.71}$$

$$\frac{\partial S}{\partial t_1} = [H(q_i, p_i, t)]_{t=t_1} , \qquad \frac{\partial S}{\partial q_{1i}} = -[p_i]_{t=t_1} , \tag{13.72}$$

in which the p_i refer to the extremal C at P_1 and P_2.

These equations are remarkable, since they enable us, if we know the function $S(q_1, t_1, q_2, t_2)$ to determine all the extremals (in mechanical terms, all the possible motions of the system) – without solving any differential equations! For suppose we are given the initial conditions (q_1, p_1, t_1), (i.e., in mechanical terms, the configuration and canonical momenta at time t_1), and also the function S. The n equations $\partial S/\partial q_1 = -p_1$ in (13.72) relate the $n + 1$ quantities (q_2, t_2) to the given constants q_1, p_1, t_1. So in principle, we can solve these equations by a purely algebraic process, to get q_2 as a function of t_2 and the constants q_1, p_1, t_1. Finally, we can get p_2 from the n equations $p_2 = \partial S/\partial q_2$ in (13.71). So indeed the extremals are found without performing integrations, i.e., just by differentiation and elimination: a very remarkable technique.[10]

Substituting the second set of equations of (13.71) in the first yields

$$\frac{\partial S}{\partial t_2} + H(q_2, \partial S/\partial q_2, t_2) = 0 . \tag{13.73}$$

[10] As Hamilton of course realized. He writes, in the impersonal style of the day: "Mr Hamilton's function S ... must not be confounded with that so beautifully conceived by Lagrange for the more simple and elegant expression of the known differential equations [i.e., L]. Lagrange's function states, Mr Hamilton's function would solve the problem. The one serves to form the differential equations of motion, the other would give their integrals" (1834, p. 514).

So the characteristic function $S(q_1, t_1; q_2, t_2)$ considered as a function of the $n + 1$ arguments $(q_2, t_2) = (q_{2i}, t_2)$ [i.e., with (q_1, t_1) fixed] satisfies the Hamilton–Jacobi equation.

Assuming that this solution S is C^2, it follows from result (ii) of Sect. 13.2.2 that S defines a family of geodesically equidistant hypersurfaces, namely the geodesic hyperspheres (for short, geodesic spheres) with centre $P_1 = (q_1, t_1)$. Thus the sphere with radius R is given by the equation

$$S(q_1, t_1; q_2, t_2) = R \, , \tag{13.74}$$

with (q_1, t_1) considered fixed. So every point P_2 on this sphere is connected to the fixed centre $P_1 = (q_1, t_1)$ by a unique extremal along which the fundamental integral has value R. These extremals cut the spheres (13.74) transversally.

These geodesic spheres about the various points P_1 are special families of hypersurfaces. For by taking envelopes of these spheres, we can build up successive members of an arbitrary family of geodesically equidistant hypersurfaces. This is the basic idea of *Huygens' principle* in geometric optics. Though Huygens first stated this idea as part of his wave theory of light, it can be stated entirely generally. Indeed, there is a rich theory here. We will not enter details,[11] but just state the main idea.

Thus consider some arbitrary solution $S(q_i, t)$ of the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H(q_i, \partial S/\partial q_i, t) = 0 \, , \tag{13.75}$$

and thereby the canonical field (congruence) K belonging to it, for which $p_i = \partial S/\partial q_i$. Let h_1, h_2 be two hypersurfaces corresponding to values σ_1, σ_2 of S, i.e., $(q_i, t) \in h_j$, $(j = 1, 2)$ iff $S(q_i, t) = \sigma_j$. Let P_1 be in h_1, and let the canonical extremal C through P_1 intersect h_2 in P_2. Then we already know from (13.19) that the fundamental integral along C is

$$\int_{P_1}^{P_2} L \mathrm{d}t = \sigma_2 - \sigma_1 \, , \tag{13.76}$$

so that P_2 is in the geodesic sphere centred on P_1 with radius $\sigma_2 - \sigma_1$. Huygens' principle states that more is true: h_2 is the envelope of the set of geodesic spheres of radius $\sigma_2 - \sigma_1$ with centres on the hypersurface h_1.

As a final task for this section, we briefly illustrate our formalism with another topic in geometric optics: namely, Fermat's least time principle, which states (roughly speaking) that a light ray between spatial points P_1 and P_2 travels by the path that makes stationary the time taken. This illustration has two motivations. First, together with the discussion in the next section, it

[11] For details, see Baker and Copson (1950) and Herzberger (1958). In optics, the Hamilton–Jacobi equation is often called the *eikonal* equation.

will bring out the optico-mechanical analogy – and so prompt the transition to wave mechanics.

Second, it illustrates how our formalism allows t to be a coordinate like the q_i, even though it is singled out as the integration variable (see Sect. 13.5). In fact, there are subtleties here. For if one expresses Fermat's principle using time as the integration variable, one is led to an integrand that is in general, e.g., for isotropic media, homogeneous of degree 1 in the velocities $\dot{q}_i$; and as noted in the second remark right at the end of Sect. 13.2.1, this conflicts with our requirement of a non-vanishing Hessian (13.4), i.e., with our construction of a canonical formalism. So illustrating our formalism with Fermat's principle in fact depends on using a spatial coordinate as integration variable (parameter along the light path). As we will see in a moment, this gives an integrand which is in general, even for isotropic media, not homogeneous of degree 1 in the velocities, so that we can apply the theory of Sect. 13.2 onwards.

So now our preferred coordinate t will be (not time, as it will be in mechanics) but one of just three spatial coordinates (q_1, q_2, t) for ordinary Euclidean space. In fact, applications of geometric optics, e.g., to optical instruments which typically have an axis of symmetry, often suggest a natural choice of the coordinate t.

At a point $P = (q_1, q_2, t)$ in an optical medium, a direction is given by direction ratios $(\dot{q}_1, \dot{q}_2, \dot{t}) = (\dot{q}_1, \dot{q}_2, 1)$. (So note that subscripts 1 and 2 now refer to the first and second of three spatial axes 'at a single time', and not to initial and final configurations.) The speed of a ray of light through P in this direction will in general depend on both position and direction, i.e., on the five variables $(q_i, \dot{q}_i, t)$, $i = 1, 2$, and so the speed is denoted by $v(q_i, \dot{q}_i, t)$. If c is the speed of light in vacuo, the *refractive index* is defined by

$$n(q_i, \dot{q}_i, t) := c/v(q_i, \dot{q}_i, t) \; . \tag{13.77}$$

If n is independent of the directional arguments $\dot{q}_i$ (respectively, positional arguments q_i, t), the medium is called *isotropic* (respectively, *homogeneous*).

Now let the curve $C : q_i = q_i(t)$ represent the path of a light ray between two points P_1, P_2 with parameter values $t = t_1, t = t_2$. Then the time taken to traverse this curve (the *optical length* of the curve) is

$$T = \int_{t_1}^{t_2} \frac{\mathrm{d}s}{v} = \int_{t_1}^{t_2} \frac{n(q_i, \dot{q}_i, t)}{c} \left[(\dot{q}_1)^2 + (\dot{q}_2)^2 + 1\right]^{1/2} \mathrm{d}t = \int_{t_1}^{t_2} L \, \mathrm{d}t \; , \tag{13.78}$$

where we have defined

$$L(q_i, \dot{q}_i, t) := \frac{n(q_i, \dot{q}_i, t)}{c} \left[(\dot{q}_1)^2 + (\dot{q}_2)^2 + 1\right]^{1/2} . \tag{13.79}$$

However, our discussion will not be concerned with this special form of L. We will only require that L be C^2, and that the Hessian does not vanish, i.e.,

(13.4) holds. One immediately verifies that this is so for isotropic media. In fact the Hessian is

$$\frac{n(q_i,t)^2}{c^2}\left[(\dot{q}_1)^2+(\dot{q}_2)^2+1\right]^{-2} \neq 0\,.$$

We can now connect our discussion with the principles of Fermat and Huygens. We can again take Fermat's principle in the rough form above, viz., that a light ray between points P_1 and P_2 travels by the path that makes stationary the time taken. It follows that if light is instantaneously emitted from a point source located at $P_1 = (q_{1i}, t_1)$ (where now we revert to using 1 to indicate an initial location), then after a time T the light will register on a surface, $F(T)$ say, such that each point $P_2 = (q_{2i}, t_2)$ on $F(T)$ (where similarly, 2 indicates a final location) is joined to P_1 by an extremal along which the fundamental integral assumes the common value T. This surface is the *wave front* for time T, due to the point source emission from P_1. Clearly, the family of wave fronts, as T varies, is precisely the family of geodesic spheres [for L as in (13.79)] around P_1.

Using the Hamilton–Jacobi equation (13.75) (now with just three independent variables q_1, q_2, t), we can readily generalize this, so as to describe the construction of successive wave fronts, given an initial wave front. Given an arbitrary solution $S(q_1, q_2, t)$ of (13.75), and an initial hypersurface h_1 given by $S(q_i, t) = \sigma_1$, we can construct at each point $P_1 \in h_1$ the unique extremal of the canonical field belonging to the family of hypersurfaces of constant S. By Fermat's principle, each such extremal can represent a ray emitted from P_1. If we define along each such extremal the point P_2 such that the fundamental integral $\int_{P_1}^{P_2} L\,\mathrm{d}t$ attains the value T, then the locus of these points P_2 is the surface $S = \sigma_1 + T$. Thus we construct a family of geodesically equidistant hypersurfaces.[12] To sum up, each solution of the Hamilton–Jacobi equation represents a family of wave fronts, and the canonical field belonging to a family represents the corresponding light rays.

13.8 From the Optico-Mechanical Analogy to Wave Mechanics

The rise of wave optics in the nineteeth century led to geometric optics being regarded as the short-wavelength regime of a wave theory of light. So its equations and principles, such as the Hamilton–Jacobi equation and Fermat's and Huygens' principles, came to be seen as effective statements derived in

[12] The vector $p_i = \partial S/\partial q_i$ is longer the more rapidly S increases over space, i.e., the more rapidly the light's time of flight increases over space. So Hamilton called p_i the vector of normal slowness.

the short-wavelength limit of the full wave theory. But the details of these derivations are irrelevant here.[13]

For us the relevant point is that [as is often remarked, e.g., Synge (1954, Preface), Rund (1966, p. 100)] once one considers this development, together with the optico-mechanical analogy as stated so far (i.e., as it stood for Hamilton), it is natural to speculate that there might be a wave mechanics, just as there is a wave optics. That is, it is natural to speculate that classical mechanics might describe the short-wavelength regime of a wave mechanics, just as geometric optics describes the short-wavelength regime of a wave optics. This is of course precisely what de Broglie, and then Schrödinger, did. To be more specific, using our Hamilton–Jacobi perspective: they proposed that S represented, not an ensemble of systems each fully described by its classical mechanical state (q, p), but a property of an individual system.[14]

In this section, we give a simple sketch of this proposal. But we shall not give details of de Broglie's and Schrödinger's own arguments, which are subtle and complicated [Dugas (1988, Part V, Chap. 4) gives some of this history]. Our sketch is formal, though in the textbook tradition [Rund (1966, pp. 99–109) and Goldstein (1950, pp. 307–314)]; various books give fuller accounts, e.g., using the concepts of Fourier analysis and the group velocity of a wave packet, e.g., Messiah (1966, pp. 50–64), Gasiorowicz (1974, pp. 27–32)]. More precisely, we will first describe how, when we apply Hamilton–Jacobi theory to a classical mechanical system, the S-function defines for each time t surfaces of constant S in configuration space, so that by varying t we can calculate the velocity with which these 'wave fronts' propagate (in configuration space). So far, so classical. But then we will postulate that these wave fronts are surfaces of constant phase of a time-dependent complex-valued wave function on configuration space. This will lead us, with some heuristic steps, to the Schrödinger equation and so to wave mechanics.

Let us consider a classical mechanical system with holonomic ideal constraints, on which the constraints are solved so as to give an n-dimensional configuration space Q, on which the q_i are independent variables. More technically, Q is a manifold, on which the q_i are a coordinate system, and on which the kinetic energy defines a metric. But we shall not go into this as-

[13] See, e.g., Born and Wolf (1999, Sects. 3.1, 8.3.1); and Taylor (1996, Sects. 6.6–6.7) is a brief but advanced mathematical discussion.

[14] Of course, successful proposals often seem 'natural' in hindsight, and some authors [e.g., Goldstein (1950, p. 314)] maintain that de Broglie's and Schrödinger's proposal would have seemed merely idle speculation if it had been made independently of the introduction of Planck's constant and the subsequent struggles of the old quantum theory. Indeed, even in that context it was obviously both daringly imaginative (witness the fact that the S wave propagates in multi-dimensional configuration space), and confusing (witness the interpretative struggles over the reality of the wave function). In any case, whether the proposal was natural or not – after all, 'natural' is a vague word – all can now agree that their achievement was enormous.

pect. We shall simply assume that Q is equipped with the usual Euclidean metric on $\mathbb{R}^n$, and that the q_i are rectangular coordinates. We further assume that any constraints are time-independent (scleronomous), i.e., any configuration in Q is possible for the system throughout the time period in question. The result of these assumptions is that the region $G \subset \mathbb{R}^{n+1}$ for which the formalism of Sect. 13.2 has been developed is now assumed to be an 'event space' or 'extended configuration space' of the form $Q \times T$, where $T \subset \mathbb{R}$ is some real interval representing a period of time. Finally, we will assume that our system is conservative, with energy E.

Now we will presume, without rehearsing the usual equations [see especially Sect. 13.2.1 and (13.69) to (13.73)], that using the above assumptions, the Lagrangian and Hamiltonian mechanics of our system has been set up. So if $S(q_i, t) = \sigma$ is a family of geodesically equidistant hypersurfaces associated with the system (each hypersurface n-dimensional), the family covering our region G simply, then S satisfies the Hamilton–Jacobi equation in the form $\partial S/\partial t + E = 0$. This can be immediately integrated to give, for some function S^* of q_i only,

$$S(q_i, t) = S^*(q_i) - Et \; . \tag{13.80}$$

Hence the p_i of the canonical field depend only on S^*: $p_i := \partial S/\partial q_i = \partial S^*/\partial q_i$. So the hypersurfaces of our family can be written as

$$S^*(q_i) = Et + \sigma \; . \tag{13.81}$$

For any fixed t, a hypersurface of constant S, considered as a hypersurface in the configuration space Q (a hypersurface of dimension $n-1$, i.e., codimension 1), e.g., the surface $S(q_i, t) = \sigma_1$, coincides with a hypersurface of constant S^*. For this example, the surface $S^* = \sigma_1 + Et$. But while the surfaces of constant S^* are time-independent, the surfaces of constant S vary with time. So we can think of the surfaces of constant S as propagating through Q. With this picture in mind, let us calculate their velocity.

As an aside, we can state the idea of surfaces in Q of constant S more rigorously, using our assumption that the region $G \subset \mathbb{R}^{n+1}$ is of the form $Q \times T$. This implies that any equation of constant time, $t = \text{const.}$, defines an n-dimensional submanifold of G which is a 'copy' of Q. Let us call it Q_t. Each hypersurface in (13.81) defines an $(n-1)$-dimensional submanifold of Q_t (a hypersurface in Q_t of codimension 1) given by

$$S^*(q_i) = Et + \text{const.} \quad (\text{with } t \text{ constant}) \, . \tag{13.82}$$

Then, as in the previous paragraph, fixing the constant σ but letting t vary, and identifying the different copies Q_t of Q, we get a family of $(n-1)$-dimensional submanifolds of Q, parametrized by t. This can be regarded as a wave front propagating over time through the configuration space Q.

Let us now fix a constant σ and a time t. Let $P = (q_i) \in Q$ be a point on the surface $S = S^* - Et = \sigma$ and consider the normal to this surface

(pointing in the direction of propagation) at P. Then the i th component n_i of the unit normal is $n_i = |\nabla S^*|^{-1}\partial S^*/\partial q_i$. Consider a point $P' = (q_i + \mathrm{d}q_i)$ that lies a distance $\mathrm{d}s$ from P along this normal (so $\mathrm{d}q_i = n_i \mathrm{d}s$). P' is on a subsequent wave front (i.e., with the same value σ of S, but not of S^*) at time $t + \mathrm{d}t$, where by (13.81)

$$\mathrm{d}S^* = \sum_i \frac{\partial S^*}{\partial q_i}\mathrm{d}q_i = E\,\mathrm{d}t\ . \tag{13.83}$$

Dividing by $\mathrm{d}s$, this yields

$$\frac{\mathrm{d}S^*}{\mathrm{d}s} \equiv |\nabla S^*| = \sum_i \frac{\partial S^*}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}s} = E\frac{\mathrm{d}t}{\mathrm{d}s}\ . \tag{13.84}$$

But we also have

$$p_i = \frac{\partial S^*}{\partial q_i} \Longrightarrow p := |p| = |\nabla S^*|\ . \tag{13.85}$$

Combining these equations, (13.84) and (13.85), we deduce that the speed u of the wave front $S = \sigma$, i.e., $u := \mathrm{d}s/\mathrm{d}t$, is

$$u = \frac{E}{p}\ . \tag{13.86}$$

So far, so classical. But now we postulate that the wave fronts (13.81) [or (13.82)] are surfaces of constant phase of a suitable time-dependent complex-valued function ψ on Q. This postulate, together with some heuristic steps, including a judicious identification of Planck's constant, will give us a heuristic derivation of the Schrödinger equation. We will assume to begin with that we can write the postulated function $\psi = \psi(q_i, t)$ as

$$\psi = R(q_i, t)\exp\left\{-2\pi\mathrm{i}\big[\nu t - \phi(q_i)\big]\right\}\ , \tag{13.87}$$

with R and ϕ real, so that $\nu t - \phi$ is the phase, and (apart from R possibly having a t-dependence) ν is the frequency associated with ψ. Then our postulate is that there is some constant h such that

$$h\big[\nu t - \phi(q_i)\big] = Et - S^*(q_i)\ . \tag{13.88}$$

But this must hold for all q_i, t, so that

$$E = h\nu\ , \qquad S^*(q_i) = h\phi(q_i)\ . \tag{13.89}$$

So the postulated frequency is proportional to the system's energy. Then, using our previous calculation of the speed u, and the relation $u = \lambda\nu$ with λ the wavelength, we deduce that the wavelength is inversely proportional to the magnitude of the system momentum. That is,

$$u = \lambda\nu = \frac{E}{p} \Longrightarrow \lambda = \frac{h}{p} \,. \tag{13.90}$$

Substituting (13.89) in (13.87), we can write ψ as

$$\psi = R(q_i, t) \exp\left\{\frac{2\pi \mathrm{i}}{h}\big[S^*(q_i) - Et\big]\right\} = R(q_i, t) \exp\left\{\frac{\mathrm{i}}{\hbar}\big[S^*(q_i) - Et\big]\right\} \,, \tag{13.91}$$

where we have defined $\hbar := h/2\pi$.

Assuming now that R has no q_i-dependence, differentiation of (13.91) with respect to q_i yields

$$\frac{\partial \psi}{\partial q_i} = \frac{\mathrm{i}}{\hbar}\frac{\partial S^*}{\partial q_i}\psi \,. \tag{13.92}$$

Recalling that $p_i = \partial S^*/\partial q_i$, this is an eigenvalue equation, and suggests that we associate with the i th component of momentum p_i of a system whose R has no q_i-dependence the operator $\hat{p}_i$ on wave functions ψ defined by

$$\hat{p}_i := \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial q_i} \,, \qquad i = 1, \ldots, n \,. \tag{13.93}$$

Let us postulate this association also for q_i-dependent R. Then this suggests we also associate with the energy of the system the operator $\hat{H}$ on wave functions defined by

$$\hat{H} := H(q_i, \hat{p}_i, t) \,, \tag{13.94}$$

where we understand q_i, and functions of it, as operating on wave functions by ordinary multiplication.

But assuming now that R has no t-dependence, differentiation of (13.91) with respect to t yields

$$\mathrm{i}\hbar\frac{\partial \psi}{\partial t} = E\psi \,, \tag{13.95}$$

suggesting that we should associate with the energy of a system the operator $\hat{E}$ on wave functions defined by

$$\hat{E} := \mathrm{i}\hbar\frac{\partial}{\partial t} \,. \tag{13.96}$$

Incidentally, this definition is also motivated by treating time as a coordinate along with the q_i (see the discussion in Sect. 13.5). Thus (13.93) suggests that we define

$$\hat{p}_{n+1} := \frac{\hbar}{i}\frac{\partial}{\partial t} \,.$$

When this is combined with (13.96), we get

$$\hat{p}_{n+1} + \hat{E} = 0 \,, \tag{13.97}$$

which is analogous to (13.52).

If for general $R(q_i, t)$ we endorse both these suggestions – i.e., we identify the actions on (13.91) of these two suggested operators, (13.94) and (13.96) – then we get

$$\hat{H}\psi = \mathrm{i}\hbar \frac{\partial \psi}{\partial t} \,, \tag{13.98}$$

which, once we identify h as Planck's constant, is the Schrödinger equation.

13.9 A Glance at the Pilot-Wave Theory

So much by way of sketching the Hamilton–Jacobi perspective on the heuristic route to wave mechanics. In this final section, I will briefly return to the question posed by this volume: 'Quo vadis, quantum mechanics?', i.e., to the foundations of quantum theory. First, I want to stress that Hamilton–Jacobi theory remains an important ingredient in various research programmes in this field. Prominent among these is the trajectory representation of quantum mechanics, pioneered by Floyd, and Faraggi and Matone. I cannot go into details, but would recommend, as places to begin reading, both Floyd (2002) and Faraggi and Matone (2000). (Besides, Sect. 1 of the latter ends with some references to other research programmes that use Hamilton–Jacobi theory.)

I shall instead end on Hamilton–Jacobi theory in the context of another prominent research programme (related to the trajectory representation): de Broglie's and Bohm's pilot-wave theory. Again, this is a large topic, and we only wish to advertise the recent work of Holland (2001, 2001a).

First, we recall [Bohm (1952, p. 169), Bohm and Hiley (1993, p. 28), Holland (1993, pp. 69, 134)] that:

- Writing $\psi = R(q_i, t) \exp\left[\mathrm{i}S(q_i, t)/\hbar\right]$, for R, S real, in the one-particle Schrödinger equation (13.98) with $\hat{H} := (\hbar^2/2m)\nabla^2 + V$ gives

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + Q + V = 0 \,, \quad \text{with} \quad Q := \frac{-\hbar^2}{2m}\frac{\nabla^2 R}{R} \,, \tag{13.99}$$

  which looks like the classical Hamilton–Jacobi equation [see (13.23)] of a particle in an external potential that is the sum of V and Q, which Bohm called the quantum potential. Indeed Bohm and Hiley call (13.99) the quantum Hamilton–Jacobi equation. Furthermore,

$$\frac{\partial \rho}{\partial t} + \frac{1}{m}\nabla \cdot (\rho \nabla S) = 0 \,, \quad \text{with} \quad \rho := R^2 \,. \tag{13.100}$$

- These equations suggest that the quantum system comprises both a wave, propagating according to the Schrödinger equation, and a particle, which has a continuous trajectory governed by the wave according to the guidance equation

$$m\frac{\mathrm{d}q_i}{\mathrm{d}t} = \left.\frac{\partial S}{\partial q_i}\right|_{q_i=q_i(t)} , \tag{13.101}$$

  and a probability distribution given at all times by $|\psi|^2 = R^2$.

Besides, comments and equations similar to these apply when we insert $\psi = R\exp(\mathrm{i}S/\hbar)$ into the many-particle Schrödinger equation [Bohm (1952, p. 174), Bohm and Hiley (1993, p. 56 et seq.), Holland (1993, p. 277 et seq.)].

So far, so good. But Holland (2001, p. 1044) points out that the relation of pilot-wave theory to classical Hamilton–Jacobi theory is not transparent. In particular, he points out:

- The guidance law (13.101) is "something of an enigma". It looks like one half of a canonical transformation that trivializes the motion of a classical system (by transforming to a set of phase space coordinates that are constant in time). But what about the other half; and more generally, can (13.101) be somehow related to a Hamiltonian or Hamilton–Jacobi theory?
- The dependence of Q on S [through (13.100)] means that the quantum Hamilton–Jacobi equation (13.99) in effect contains higher derivatives of S – wholly unlike a classical Hamilton–Jacobi equation.

So Holland undertakes an extensive investigation of this relation. More precisely, he undertakes to formulate the pilot-wave theory as a Hamiltonian theory. He does this by assessing a treatment of Q as a field function of q_i on a par with the classical potential V, i.e., a treatment that takes as the Hamiltonian of the (one-particle) system

$$H(q_i, p_i, t) = \frac{1}{2m}\sum_i \left[p_i^2 + Q(q_i, t) + V(q_i, t)\right] . \tag{13.102}$$

He emphasises that such a treatment faces three obstacles. In brief, these are:

- As we mentioned above, Q depends on S and so presumably, by $p = \partial S/\partial q$, on p. So in a Hamiltonian (phase space) treatment, it seems wrong to take Q as a function of q alone.
- The free choice of initial positions and momenta in a Hamiltonian treatment will mean that most motions, projected on q, do not give the orthodox quantal distribution, in the way that (13.101) and $|\psi|^2 = R^2$ does.
- Is such a treatment compatible with the Hamiltonian description of the Schrödinger equation? For it to be so, we have to somehow formulate the particle–wave interaction so as to prevent a back-reaction on the wave.

However, Holland goes on to show (2001, 2001a) that these obstacles can be overcome. That is, he vindicates the proposal (13.102) with a Hamiltonian theory of the interacting wave–particle system. But we cannot enter details. It must suffice to list some features of his work. In short, his approach:

- generalizes a canonical treatment of a classical particle and associated ensemble;
- necessitates the introduction of an additional field of which the particle is the source;
- makes the quantum Hamilton–Jacobi equation and the continuity equation (13.100) (and other equations for the evolution of particle and field variables) come out as Hamilton equations;
- interprets $p = \partial S/\partial q$ as a constraint on the phase space coordinates of the wave–particle system;
- gives a general formula expressing the condition that the particle's phase space distribution, projected on q, gives the orthodox quantal distribution;
- yields a Hamilton–Jacobi theory of the wave–particle system.

To conclude, I hope to have shown that Hamilton–Jacobi theory, understood from the perspective of the calculus of variations, gives us insight into both mechanics and optics – and that, as illustrated by this last section, Hamilton–Jacobi theory is an important ingredient in current attempts to answer the question: 'Quo vadis, quantum mechanics?'

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14 Roundtable Discussion II: Quantum Mechanics and its Limits

't Hooft: I'd like to make a general comment on quantum mechanical experiments. This workshop is titled 'Quo Vadis (where are you going) Quantum Mechanics?' but an equally important question should be 'Unde venis (where do you come from) quantum mechanics?'

Unidentified [whispers]: Utrecht!

't Hooft: As for the theory's future, it's very unlikely that any breakdown of quantum mechanics follows from experiments either like these gedanken-experiments of Elitzur and Dolev's talk or Anton's experiments, which are very precise experiments about the interference of live systems. It's quite unlikely that quantum mechanics will break down in these instances. These are magnificent experiments and I think they are cleverly done. But they are like what magicians do. You know, I can be thrilled by a magician doing things which I am sure are impossible, can't be right. This apple wasn't there before and now it is there. Or this dove was not in his hat and now it is. I can't explain how it works, but I also know it is all classical physics.

Similarly, I think the real deviation from quantum mechanics, if any, should happen at extremely high energies or in extreme circumstances, like matter at extreme densities, etc. There is talk about deviations from special relativity, if you look at photons with extremely high energy which have traveled long distances in space, something like that. They see minute deviations from the laws of physics as we know them.[1] We expect possibly very, very minor deviations from quantum mechanics in these extreme circumstances. The most extreme circumstance we can think of is physics at the Planck scale, where particles collide with each other at Planckian energies, and make black holes, and so on. This is extreme physics far away from what we do even as physicists in our daily lives. So that's where you expect possible deviations.

Hiley: I want to get things clear here because I agree with you that we want to try and push at the boundaries, doing experiments to see whether

[1] M. Drees: Particle physics explanations for ultra high energy cosmic ray events, invited plenary talk at PASCOS03, Mumbai, India, January 2003. Los Alamos archives `hep-ph/0304030` (2003).

quantum theory is going to break down or not. My only issue here is whether we use the Bohm approach or standard quantum theory. Now, what does Bohm do? Bohm says,[2] all right, we have got this equation, the real part of the Schrödinger equation and with it the quantum potential. We want to have an ontological (or realist) description. If we assume particles follow trajectories, and we can calculate these trajectories from this equation then we can get an interpretation of quantum mechanics. It works and is not in conflict with experiment.

Suppose you now want to ask something like: Does the particle really follow the trajectory or does it sit on some sub-quantum medium which generates a stochastic process affecting the particle? One could test the latter possibility only if one had a good theory that produces different results from the orthodox quantum mechanics. My frustration is with people who waste time trying to distinguish between the Bohm approach and standard quantum mechanics without adding anything new. Both approaches are based on the same mathematics and will always produce the same empirical results unless something new is introduced. We need new ideas and what we should be doing is what Lee [Smolin] is doing, and what Penrose is suggesting. They are exploring new ideas that will produce different results, which if correct will show that quantum theory is *wrong*, or better still, show that it is *limited* in some way.

Hartle: Do you see any signs that Bohm theory, or quantum mechanics, breaks down as an experimentally correct description, or would you care to speculate on what scale that might happen?

Hiley: No, otherwise I'd be singing it from the rooftops! The sort of thing that Penrose proposed was something that should be encouraged.[3] The other idea was something that Prigogine was trying to work on.[4] He suggested that there was some irreversible dynamics underlying quantum theory. In other words he was actually changing the theory in some way, motivated by good physical reasons. Not changing it merely for the sake of change. That's how I would try to push it. It is difficult to produce something new that works but we should keep on trying rather than keeping the old game of 'whose

[2] D. Bohm and B.J. Hiley: *The Undivided Universe: An Ontological Interpretation of Quantum Theory*, Routledge, London (1993).

[3] The allusion is to Penrose's hypothesis that quantum superpositions remain as long as the difference in the spacetime manifold, entailed by the different mass configurations, is kept below some threshold value. Above that limit, he proposed, some nonlinear dynamics, emanating from an as yet unknown theory of quantum gravity, will cause a collapse. See, e.g., R. Penrose: Non-locality and objectivity in quantum state reduction. In: *Fundamental Aspects of Quantum Theory*, ed. by J. Anandan and J.L. Safko, World Scientific, Singapore (1994).

[4] I. Prigogine: The Arrow of Time. In: *The Chaotic Universe*, ed. by V.G. Gurzadyan and R. Ruffini, World Scientific, Singapore (2000).

interpretation is right' going. I think all interpretations are limited. All they are doing is talking in different ways about the same formalism. We want some new formalism. Even Gerard ['t Hooft]'s work will be suggesting things which will lead to different results and that's what we should be looking for.

Hartle: Am I correctly interpreting the subject of discussion as: Where would quantum theory break down?

Smolin: One thing which people have said for a long time, which turns out to be wrong, is that there is no possibility of sensitivity to the Planck scale. Many people have realized in the last two or three years that experiments with cosmic rays, experiments with gamma-ray bursts, give us sensitivity to breakdown of symmetries, for example the Lorentz invariance, at the scale of the energy of the particles times the Planck length. These are real experiments, for example, the GLAST (Gamma Ray Large Area Space Telescope) experiment, will measure energy dependence of photons to the order of energy times the Planck length. And there are also very sensitive experiments regarding CPT breaking still going on. And one should look for these kinds of things. I draw two conclusions from that. First, all the alternative proposals about changes to quantum mechanics which I mentioned and then discarded because they can't be made Lorentz invariant, for which there is a list, maybe should not be so quickly thrown away, because maybe Lorentz invariance is not a real symmetry of nature, and certainly a discovery of breakdown of Lorentz invariance is quite within the realm of possibilities. There are people who claim that violations of the GZK cutoff found in cosmic rays[5] already signal breakdown of Lorentz invariance. Another thing is that we should never underestimate the imagination and the skill of the experimenters to surprise us, and I think to become complacent about what will not happen soon experimentally is silly, because one only has to go back a hundred years and read the very complacent statements by Mach and other people that atoms would never be observed and therefore certain theoretical avenues should be closed off. It's really shocking how silly theorists can get about what experimentalists won't be able to do!

Saunders: I want to remark about the pilot-wave theory. It seems to me that there is a view here that it is equivalent to standard quantum mechanics. But no one to this day produced a single model of pair creation or pair annihilation events using the pilot wave formalism. The pilot wave theory exists as a non-relativistic theory alone.

[5] K. Greisen: Phys Rev. Lett. **16**, 748 (1966); G.T. Zatsepin and V.A. Kuzmin: JET Ph. Lett. **4**, 78 (1966).

Hartle: Can't pair creation be handled in Bohm theory by using quantum fields as variables?

Saunders: No, because you're going to use field configurations as the 'beables',[6] and you need a recipe for translating trajectories of field configurations into pair creation and annihilation events. And furthermore you need to show that things like measurement pointer positions are well defined, and well localized in space. There has not been a single model produced of that form, and nor has a single argument shown that you could expect that such a model will do the job. The only reason that standard pilot wave theory gives you well localized events is because it puts in particles as beables. Once you start putting field configurations as these beables, there is as yet not even an intuitive hand-waving argument to show or to give any reason to think that you will get well localized events where you need them.

Hartle: Basil, do you want to comment on that?

Hiley: I agree with it. One of the problems that David [Bohm] and I had was how the classical limit emerges from the pilot wave theory as we were working on it. It emerges once you have a localized wave packet. But our problem was how you localize the wave packet in the first place when starting from the field theoretic viewpoint.

Stapp: As I see the problem, the extra part of the Bohm theory that is important is of course that this trajectory has certain *empirical* consequences. When this trajectory passes through a detector, that detector is supposed to fire. When the observer is watching a counter, if the trajectory is in the channel that is passing through that detector, then that detector ought to fire. So Elitzur (see Chap. 17) is pointing out a situation where we have several counters in a row and presuming one trajectory is either passing through all of them or none of them. So how can you explain in a Bohm theory that just the central detector fires and the others on the outside do not fire?

Hiley: First of all I would need notice of that question. But my comment would be that this is a more elaborate version of the Scully objection.[7] The detectors we are trying to fool are still quantum systems. And if they are still quantum systems they don't constitute an actual measurement where something irreversible has definitely happened. This means that the quantum potential is still active, but it is non-local, and it's through the non-local quantum potential that you account for these phenomena. Now we might

[6] The speaker refers here to a useful term coined by J.S. Bell in his *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, 1987), where he distinguished between 'beables' and 'speakables'.

[7] B. Scully: Physica Scripta **T76**, 41–46 (1998).

argue that we don't like nonlocality, and the type of experiments Elitzur and Dolev are doing makes this nonlocality worse and worse. Then you could say: "All right, I don't want to follow this idea any more, and I'll go back to the Copenhagen view." But I don't see that we've reached that particular level yet.[8]

Hartle: If these seven objects in a row really were detectors, would you agree with Elitzur and Dolev? That is, if you looked at them and saw that one out of the row had fired and not the others, would you agree that there is a problem?

Hiley: No, I am not saying there would be a problem.

Elitzur: Not even then? Even if they were classical detectors?

Hiley: No. The reason why you think you are beating the Bohm interpretation is because you are using atoms instead of classical systems.

Butterfield: I think there is a bit of confusion here because of the word 'detector'. When Henry [Stapp] says "if it's really detecting", I think he means something like "if you know for sure that it can't be fooled". But the point of the Hiley escape from the Scully kind of puzzle is that, as I understand it, we must not think that in the Bohm theory position measurement is always, as the philosophers call it, veridical. It doesn't always speak the truth. Position is favored as a quantity in the theory, but detectors can be fooled.

Hiley: Yes, and there is an explanation of why the fooling takes place, through the quantum potential and its non-locality.

Rovelli: I'd like to say something about the breakdown of quantum mechanics. I've heard a lot of comments on where we expect, or search for this breakdown. Of course, in fundamental theoretical physics everything has changed dramatically and may well continue to change forever. But we have discovered quantum mechanics and it's also possible that quantum mechanics is going to last as long as classical mechanics. And if this is so – and I believe it is – we have to get used to it and understand it. When Galileo discovered the Galilean invariance, and the main idea that is a part of this view, namely that there is no center of the universe, it took forever for people to just get used to that, to the point that, if we read the *Principia*, there is a nice passage in which Newton says: "Yes, of course, but there should be a center of the universe which is not the sun, but it should be close to the sun" – which is total nonsense. This is a point in which Newton was going silly – even him – because all the rest of his theory doesn't need a preferred reference system,

[8] I have now shown that non-locality is not necessary to answer Scully's objections. (BH)

and doesn't need a center of the universe. So even he couldn't get used to the new philosophy that had come out. We should try to extract from quantum mechanics and general relativity what these extremely subversive and radical theories are trying to tell us. They give us extremely radical mental frameworks for understanding the world. The big challenge is not to be so smart to look for the breakdown of quantum mechanics. The challenge here is to get completely used to this new framework and learn about how to think of the world in terms which are completely general relativistic and completely quantum mechanical. Perhaps we have to stay with this theory for a long time.

Butterfield: I'd like to hear more from Gerard ['t Hooft] about his idea that time is a matter of the clear prescription of order with which we apply the formalism. Because that was on his projection, but it doesn't sound like anything connected with the notion of time in either orthodox physics or the normal debates we have about time in quantum gravity, such as quantum general relativity. Gerard, would you like to say more about that?

't Hooft: The idea is that time in conventional physics is, of course, also the fourth coordinate in the Minkowski space. It is also the coordinate which plays a central role in Schrödinger's equation. And in GR time is again basically the fourth coordinate which has a negative eigenvalue or the different eigenvalue in the metric tensor. And all these notions of time coincide. That doesn't have to be so, but I think the most important aspect of time is, of course, causality, the fact that things in the past are known and things in the future appear to be determined by things happening in the past, at least according to classical physics. I would be in favor of the idea that the same thing should still hold also for quantum mechanics, so that the future is determined by the past according to some principle of causality. I'd make causality very central to my understanding of physics. If a theory were acausal, I would have a very deep philosophical difficulty with accepting it because the theory would not tell us in which order to apply its equations. I wouldn't know where to begin. If I had a closed timelike loop I wouldn't know where to start writing the history. It's always the complaint we have in science fiction stories where a closed timelike loop occurs: they begin somewhere but they don't begin at the beginning, because there is no beginning! There is no correct prescription about how to proceed. Now I don't care much about science fiction stories, but I *do* care about physical theories. I want the theory to give me a unique prescription which tells me once and for all where to start doing my calculations and how to proceed, and in which order, such that if you take the wrong order you might get the wrong answer. I want a theory which gives me one precisely guaranteed answer – not two or zero answers, but exactly one! Which means that you must give, in your prescription, the order in which you do your calculations. Now it seems a practical feature that we have gotten used to in physics that that order happens to be also the

order of the fourth coordinate of Minkowski space and the time parameter in Schrödinger's equation. That doesn't have to be so. I could imagine a deviation from quantum mechanics or a deviation from relativity where causal time no longer coincides with Minkowski time. But the fact that it appears to do so is very convenient for our understanding of the theory. If it weren't like that, we would certainly have big difficulties understanding how to proceed. But taking the notion which I have of time – that there should be something which has a causality in it – is to me by far a superior definition of time. This causality would be in my definition of time. Everything else would depend on that. If you then have a Schrödinger equation, you always know what to do. If you have a metrical spacetime again you'd always be able to solve these elliptic equations because you can write down Cauchy surfaces and the like. So if a theory deviates from that, I would want to know how and why, and how to restore causality. But in any case I think that the definition I gave of time is one which, by construction, will never allow for closed timelike loops.

Saunders: I'd like to make a comment on the three philosophies of time that were mentioned in some of the preceding lectures. And I want to ask Lee [Smolin] how Carlo [Rovelli] gives this fourth philosophy of time. But perhaps it would help if I just summarize the three philosophies. The first is the Block Universe, where all times have the same degree of reality. The second is Presentism, where only the present is real. The third is Meta-Time, where time itself is somehow evolving by some higher time parameter. It seems to me that the Block Universe is not denying that there is becoming. It rather depicts becoming. It depicts change in terms of relations among its parts. That would be just a comment on this first option. But over to Lee and Carlo's fourth alternative.

Smolin: One slogan about this which actually came from Freeman Dyson, quoting somebody else whom I don't remember, is that the classical description is the description that we give of our past and the quantum description is the description that we give of our future. And I think that when Freeman said that, and we can credit him with this, he was really taking on relational quantum theory because he was saying that one divides the world into classical and quantum realms differently according to where they are in time. And I think that's what I really want to bring up. Futini Markopoulou[9] makes this explicit in this formalism called quantum causal histories, where to every causal past of an observer there is one of Carlo [Rovelli]'s relational Hilbert spaces, and you can play Carlo's game just using the causal structure to keep track of who transmits information to whom, and so here there is no Block Universe and it's something between Becoming and Presentism and so forth. But the point is that, in order to think this way, you have to think Carlo's

[9] F. Markopoulou: Quantum causal histories, Class. Quant. Grav. **17**, 2059–2072 (2000).

way. How I divide the world into past and present is related to how I divide the world into classical and quantum. One way that I think about it – although I don't know how far this goes but take on board all of these things – is that the classical description is the description that I give to my causal past, where I can describe in classical terms what happened. Also, I can talk about my causal future in classical terms because of this weird thing that I am able to plan some of the things I will do, like decide what experiments I make. It seems that the quantum description, in terms of the quantum state and uncertainty principle and so forth, is the description I give to that which is *spacelike* to me at any moment. And a measurement is nothing special, just when some information that is now spacelike to me comes into my past. And that is 'what the measurement is'. And this idea then tells us that straightening out quantum mechanics has a lot to do with causal structure and therefore with the structure of spacetime. This is roughly this fourth alternative.

15 New Insight into Quantum Entanglement Using Weak Values

Yakir Aharonov and Shahar Dolev

Using the two-vector formalism, we show how some recent quantum mechanical paradoxes get a natural explanation and how surprising new predictions can be derived from quantum theory.

15.1 Entanglement and Pre- and Post-Selection

According to classical physics, if we know the state of a system of particles we also know the state of each individual particle. Furthermore, knowledge of this state allows us to know, at least in principle, a unique state of the particles for all times. However, this is not the case according to quantum mechanics.

First, while it is possible to know exactly the quantum state for a system of particles, this knowledge may not be enough to determine also the quantum state of the individual particles. The classic example is the singlet state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|+\rangle_1|-\rangle_2 - \frac{1}{\sqrt{2}}|-\rangle_1|+\rangle_2 \,. \tag{15.1}$$

Here, the state $|\psi\rangle$ tells us absolutely nothing about the state of particles 1 or 2 separately. Such states, which cannot be brought to a product form, i.e.,

$$|\psi\rangle = |a\rangle_1|b\rangle_2 \,, \tag{15.2}$$

are the so-called *entangled* states. Very interesting consequences related to non-locality stem from properties of these states, as in the EPR experiment.

Secondly, quantum mechanics is not deterministic. If we know the state $|\psi\rangle$ now, we cannot be sure of what the state will be later, for example after a second measurement has been performed on the system. At any given time, we should therefore 'tag' a system not only by the initial quantum state $|\psi\rangle$, but also by assigning to it a final state $|\phi\rangle$, representing some 'exit' conditions that cannot be determined from prior knowledge of $|\psi\rangle$. In fact, experiments where the system is delimited both by initial and final conditions are not uncommon. These are the so-called exclusive measurements. When a system is labeled in this way, we say that it belongs to a *pre- and post-selected* ensemble.

Based on this idea of pre- and post-selection, a whole interpretation of quantum mechanics has been developed which is called the two-vector formulation [1–3]. The idea is that, at any given time, the state of the system is described both by the initial vector $|\psi\rangle$ and the final vector $|\phi\rangle$. This new way of thinking also brings some novel insights into some of the puzzles entailed by quantum mechanical entanglement. One of the side branches of this interpretation, the weak measurement, gave such an insight [2, 4, 5].

15.2 Weak Measurements

Suppose we are given a physical ensemble of $N \gg 1$ uncorrelated spin-1/2 particles, and we are told that all of them are prepared in the same state $|\psi\rangle = |\hat{n}+\rangle$, where $\hat{n}$ is some direction unknown to us. We wish to find this direction.

One possibility is to divide the ensemble into three subensembles of size $N/3$, and within each one measure the same direction on every spin, say σ_x in the first subensemble, σ_y in the second one, and σ_z in the third. From these measurements one can obtain the expectation value $\hat{n} = \langle\psi|\boldsymbol{\phi}|\psi\rangle$ to an accuracy of order $1/\sqrt{N}$. The only problem with this method is that the ensemble we are left with will be very different from the initial one, as the particles are now in any one of the states $|\hat{x}\pm\rangle$, $|\hat{y}\pm\rangle$ or $|\hat{z}\pm\rangle$, not the initial state $|\hat{n}+\rangle$.

There is, however, an alternative type of measurement, called 'weak measurement', which allows us to find the direction $\hat{n}$ by minimally disturbing the initial state of the N-particle system.

Let us look at the composite state

$$|\psi^{(N)}\rangle = |\psi\rangle_1 |\psi\rangle_2 \ldots |\psi\rangle_N \,. \tag{15.3}$$

Instead of measuring the spin components of each system individually, we make just three successive measurements of the 'average' spin operators $\hat{\sigma}_x^{(N)}$, $\hat{\sigma}_y^{(N)}$, $\hat{\sigma}_z^{(N)}$ of the N-particle system, where we define

$$\hat{A}^{(N)} \equiv \frac{1}{N} \sum_{i=1}^{N} \hat{A}_i \,, \tag{15.4}$$

and $\hat{A}_i$ is the same operator as $\hat{A}$ acting on the ith particle.

In fact, as $N \to \infty$, the 'average' operators behave very much like classical variables, and so weak measurements are nothing more than 'classical' measurements at the macroscopic scale. To see this, suppose that

$$\left[\hat{A}, \hat{B}\right] = \mathrm{i}\hat{C} \,. \tag{15.5}$$

Then, using the fact that operators acting on different particles commute, one can show that

$$\left[\hat{A}^{(N)}, \hat{B}^{(N)}\right] = \mathrm{i}\frac{1}{N}\hat{C}^{(N)} \ . \tag{15.6}$$

Therefore, $\hat{A}^{(N)}$ and $\hat{B}^{(N)}$ commute in the limit $N \to \infty$.

More precisely, we can show that, in the same limit, any product state such as $|\psi^{(N)}\rangle$ becomes an eigenstate of the operator $\hat{A}^{(N)}$. For this we make use of a very simple but powerful identity:

$$\hat{A}|\psi\rangle = \bar{A}|\psi\rangle + \Delta A|\psi_\perp\rangle \ , \tag{15.7}$$

where $\bar{A}$ is the expectation value

$$\bar{A} = \langle\psi|\hat{A}|\psi\rangle \ , \tag{15.8}$$

ΔA is the difference

$$(\Delta A)^2 = \langle\psi|(\hat{A} - \bar{A})^2|\psi\rangle \ , \tag{15.9}$$

and $|\psi_\perp\rangle$ is some orthogonal state such that

$$\langle\psi|\psi_\perp\rangle = 0 \ . \tag{15.10}$$

We now apply this identity to $\hat{A}^{(N)}$ and $|\psi^{(N)}\rangle$. The state $\hat{A}^{(N)}|\psi^{(N)}\rangle$ is now

$$\hat{A}^{(N)}|\psi^{(N)}\rangle = \frac{1}{N}\left[N\bar{A}|\psi^{(N)}\rangle + \Delta A\sum_i |\psi_\perp^{(N)}(i)\rangle\right] \ , \tag{15.11}$$

where the states $|\psi_\perp^{(N)}(i)\rangle$ are mutually orthogonal, and are given by

$$|\psi_\perp^{(N)}(i)\rangle = |\psi\rangle_1|\psi\rangle_2 \ldots |\psi_\perp\rangle_i \ldots |\psi\rangle_N \ . \tag{15.12}$$

If we further define a normalized state

$$|\psi^{(N)}\ \perp\rangle = \sum_i \frac{1}{\sqrt{N}}|\psi_\perp^{(N)}(i)\rangle \ , \tag{15.13}$$

we obtain

$$\hat{A}^{(N)}|\psi^{(N)}\rangle = \bar{A}|\psi^{(N)}\rangle + \frac{\Delta A}{\sqrt{N}}|\psi^{(N)}\ \perp\rangle \ . \tag{15.14}$$

Therefore, $|\psi^{(N)}\rangle$ becomes an eigenstate of $\hat{A}^{(N)}$ with eigenvalue $\bar{A}$, as $N \to \infty$. We should note that the $O(N^{-1/2})$ behavior of $\Delta A^{(N)}$ holds even when the particles are not all in the same state, as long as the composite N-particle state is a product state.

So we see that it is indeed possible to arrange things, by taking N sufficiently large, in such a way that $\hat{n} = \langle\psi|\boldsymbol{\sigma}|\psi\rangle$ can be determined from just three measurements *while barely disturbing the state of the spins.*

15.3 Weak Values

Now, as we saw earlier, quantum mechanics allows us to impose additional final conditions on the system which are not encoded in the initial state vector $|\psi\rangle$. Based on this idea, a new physical quantity was proposed, the 'weak value' of a given observable. As we shall see, weak values provide a natural description of a system confined by two boundary conditions.

In order to motivate the definition of the weak value, let us first re-express the standard expectation value $\bar{A}$ by expanding it in a complete set of states $|\phi_n\rangle$:

$$\bar{A} = \langle\psi|\hat{A}|\psi\rangle = \langle\psi|\sum_n |\phi_n\rangle\langle\phi_n|\hat{A}|\psi\rangle = \sum_n |\langle\psi|\phi_n\rangle|^2 \frac{\langle\phi_n|\hat{A}|\psi\rangle}{\langle\phi_n|\psi\rangle} \ . \quad (15.15)$$

If we think of the states $|\phi_n\rangle$ as the possible outcomes of some final measurement on the system, then the coefficients $|\langle\psi|\phi_n\rangle|^2$ give us the probabilities $P(n)$ of these outcomes. We can therefore view $\bar{A}$ as the average

$$\bar{A} = \sum_n P(n) A_{\rm w}(n) \ , \quad (15.16)$$

where

$$A_{\rm w}(n) \equiv \frac{\langle\phi_n|\hat{A}|\psi\rangle}{\langle\phi_n|\psi\rangle} \quad (15.17)$$

is the weak value of $\hat{A}$ when post-selecting $|\phi_n\rangle$. In this way, we obtain an alternative interpretation of the expectation value, as an average of weak values.

In fact, weak values, as well as the decomposition (15.16), actually reflect definite outcomes of measurements that could be performed in the laboratory. These are the weak measurements discussed earlier, but performed on a pre- and post-selected ensemble. Suppose we start with N particles, all in the same initial state $|\psi\rangle$. We then perform the measurement of the 'average' observable $\hat{A}^{(N)}$. Finally the particles are post-selected by performing an ordinary measurement on each particle. If, by chance, all the particles end up in the *same* final state $|\phi_n\rangle$, the claim is then that, as $N \to \infty$, the result of the intermediate measurement must yield, with increasing certainty, the weak value

$$A_{\rm w}(n) = \frac{\langle\phi_n|\hat{A}|\psi\rangle}{\langle\phi_n|\psi\rangle} \ .$$

More precisely, suppose that the weak measuring apparatus is described by two canonically conjugate variables q and p, where p plays the role of the 'pointer' variable. The measuring interaction corresponds to a brief coupling

between the system and the apparatus that implements a translation of the operator $\hat{p}$ by $\hat{A}^{(N)}$:

$$\hat{p} \to \hat{p} + \hat{A}^{(N)} \ . \tag{15.18}$$

We now concentrate on the ensemble of N identical systems defined by the initial product state $|\psi^{(N)}\rangle$ and the final product state $|\phi^{(N)}\rangle$. Clearly, as $N \to \infty$, the probability $|\langle\phi^{(N)}|\psi^{(N)}\rangle|^2$ of obtaining the state $|\phi\rangle$ on each system goes to zero as well. However, suppose that this rare final condition does indeed occur for a finite but very large N. It has been shown (see [2] for details) that if the initial state $|\xi_i\rangle$ of the measuring apparatus has for instance a Gaussian wave function

$$\xi_i(p) \propto \exp\left[-(\Delta q)^2 p^2\right] \ , \tag{15.19}$$

then for any fixed Δq, the wave function after the interaction becomes

$$\xi_f(p) \propto \exp\left[-(\Delta q)^2 \left(p - \frac{\langle\phi|\hat{A}|\psi\rangle}{\langle\phi|\psi\rangle}\right)^2\right] \ , \tag{15.20}$$

with increasing accuracy as $N \to \infty$.

Note that the weak value $A_{\rm w}$ need not be a real number. In such a case, its real part is therefore obtained ($\pm\Delta p \sim 1/\Delta q$) by looking at the translation of the 'pointer' variable of the measuring apparatus, while the imaginary part can be obtained by looking instead at the conjugate variable q.

The weak value can also be realized statistically, by performing separate measurements of $\hat{A}$ on each of the individual systems, if in each measurement device the uncertainty Δq, which determines the disturbance, is sufficiently small. Since this also implies that the individual pointer uncertainty Δp will be large, the weak value is again obtained at the level of the ensemble, but now from the mean reading of p (to accuracy $\Delta p/\sqrt{N}$). In this sense we can reproduce the decomposition (15.16). If the final basis is $\{|\phi_n\rangle\}$, then an initial ensemble prepared in the state $|\psi\rangle$ breaks up into sub-ensembles described by the states $|\phi_n\rangle$, each of relative size $P(n) = |\langle\phi_n|\psi\rangle|^2$. The mean reading from each of these sub-ensembles will be the respective weak value

$$A_{\rm w}(n) = \frac{\langle\phi_n|\hat{A}|\psi\rangle}{\langle\phi_n|\psi\rangle} \ .$$

If we then pool all the readings, the average reading from the initial ensemble will be $\sum_n P(n)A_{\rm w}(n)$. Equation (15.16) shows that, as expected, this is nothing more than the standard average value $\langle\psi|\hat{A}|\psi\rangle$.

So we can see that weak measurements naturally define a physical quantity that encodes the properties of a system described by initial and final boundary conditions. More interestingly, these properties may differ dramatically from what we are led to expect from our quantum intuitions, based on initial conditions only:

(a) *The weak value of a sum of operators is the sum of their weak values.* Measuring the weak value of the operator $\hat{C} = \hat{A} + \hat{B}$ will yield $C_{\rm w} = A_{\rm w} + B_{\rm w}$:

$$\begin{aligned} C_{\rm w} &= \frac{\langle\phi_n|\hat{C}|\psi\rangle}{\langle\phi_n|\psi\rangle} \\ &= \frac{\langle\phi_n|\hat{A}+\hat{B}|\psi\rangle}{\langle\phi_n|\psi\rangle} \\ &= \frac{\langle\phi_n|\hat{A}|\psi\rangle + \langle\phi_n|\hat{B}|\psi\rangle}{\langle\phi_n|\psi\rangle} \\ &= A_{\rm w} + B_{\rm w} \ . \end{aligned} \tag{15.21}$$

(b) As a result, *the weak values might lie outside the range of the eigenvalues of the operator.* Suppose we start with a system of spins all prepared in the $|\sigma_x = +1\rangle$ state and post-selected in the final state $|\sigma_y = +1\rangle$. Given this unlikely event (for $N \gg 1$), a weak measurement of the spin component $\sigma_u = \boldsymbol{\sigma}\cdot\hat{u}$, where

$$\hat{u} = \frac{1}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{y} \ , \tag{15.22}$$

yields a weak value which lies outside the bounds of the spectrum of σ_u [or $\sigma_u^{(N)}$ for that matter]:

$$\frac{\langle\sigma_y = +1|\frac{1}{\sqrt{2}}\hat{\sigma}_x + \frac{1}{\sqrt{2}}\hat{\sigma}_y|\sigma_x = +1\rangle}{\langle\sigma_y = +1|\sigma_x = +1\rangle} = \sqrt{2} \quad (!) \tag{15.23}$$

Note also that in this case the weak value of σ_u is obtained from the vector sum of the weak values along $\hat{x}$ and $\hat{y}$, as in classical physics.

(c) If between the pre- and post-selection events, a standard measurement of an operator $\hat{A}$ is performed, and if $\hat{A}$ has only 2 eigenvalues (e.g., projection oprator or spin-1/2 measurement), and the result of the measurement complies with the post-selection condition, and no other measurement is performed in that interval, then *the result of the measurement of $\hat{A}$ will equal the weak value $A_{\rm w}$ of the same operator.*
That is, if the result of the measurement is

$$\hat{A}|\psi\rangle \longrightarrow a|\phi\rangle \ , \tag{15.24}$$

Then the weak value between the pre- and post-selection events is

$$A_{\rm w} = a \ . \tag{15.25}$$

(d) One of the consequences of the above result is that *weak values of non-commuting variables can be measured simultaneously.* For two non-commuting operators $\hat{A}$ and $\hat{B}$, it is possible to measure both weak values A_{w} and B_{w} between the pre- and post-selection events. The results will be the same as if each measurement were conducted separately.
This outcome is very interesting since it deals with the way counterfactual measurements are handled. In standard analysis, counterfactual measurements are mutually exclusive: between the pre- and post-selection events, measurement of one operator, $\hat{A}$, interferes with measurement of the other, $\hat{B}$. With weak measurements, in contrast, both values, A_{w} and B_{w}, can be measured simultaneously since weak measurement only slightly alters the state of the system. According to outcome (c) above, this means that, between the pre- and post-selection events, measurement of $\hat{A}$ will yield the value A_{w} while a counterfactual measurement of $\hat{B}$ would have yielded the value B_{w}.

15.4 Interaction-Free Measurements

Before seeing what all this has to do with entanglement, we will need to discuss the concept of interaction-free measurements, first introduced by Elitzur and Vaidman [6]. In the original version, the idea is as follows.

We consider a Mach–Zender Interferometer (MZI) as shown in Fig. 15.1. A photon enters the interferometer at a beam-splitter BS_1, from which it is transmitted or reflected with probability 1/2. The beam is reconverged at another 50/50 beam-splitter BS_2 by two totally reflecting mirrors M_1 and M_2, each one along one of the arms of the MZI. Since the beam suffers a rotation of π in its phase each time it is reflected, one can arrange the arms of the interferometer in such a way that the transmitted amplitude for the path,

$$\text{source} \longrightarrow \mathrm{BS}_1 \longrightarrow \mathrm{M}_2 \longrightarrow \mathrm{BS}_2 \rightarrow \mathrm{D}\ , \tag{15.26}$$

is 180° out of phase with the reflected amplitude

$$\text{source} \longrightarrow \mathrm{BS}_1 \longrightarrow \mathrm{M}_1 \longrightarrow \mathrm{BS}_2 \longrightarrow \mathrm{D}\ . \tag{15.27}$$

Since the two paths reaching detector D suffer a complete destructive interference, while the two paths that go to detector C give rise to constructive interference, the photon can only be detected by detector C.

Technically, the photon was split by BS_1:

$$|\Psi\rangle = |\gamma\rangle \xrightarrow{\mathrm{BS}_1} \frac{1}{\sqrt{2}}(\mathrm{i}|u\rangle + |v\rangle)\ . \tag{15.28}$$

The multiplication of $|u\rangle$ by i is due to phase rotation resulting from the reflection. BS_2 then brings about the following transformation:

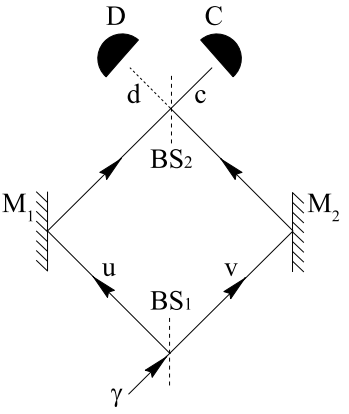

Fig. 15.1. Mach–Zhender interferometer (MZI)

$$|v\rangle \xrightarrow{\mathrm{BS}_2} \frac{1}{\sqrt{2}}(|d\rangle + \mathrm{i}|c\rangle) \,, \tag{15.29}$$

$$|u\rangle \xrightarrow{\mathrm{BS}_2} \frac{1}{\sqrt{2}}(|c\rangle + \mathrm{i}|d\rangle) \,, \tag{15.30}$$

leading to the result

$$|\Psi\rangle \xrightarrow{\mathrm{BS}_2} |c\rangle \,. \tag{15.31}$$

As demonstrated by Elitzur and Vaidman (EV) [6], IFM is achieved when a macroscopic detector is placed on path v – a super-sensitive bomb in EV's article (see Fig. 15.2). The fact that a detector is present on path v can result in one of two outcomes:

1. The detector registers a particle (or the bomb explodes), meaning that the photon went through path v and was detected there. In that case, the wave function becomes:

$$|\Psi\rangle \xrightarrow{\text{detection}} \frac{1}{\sqrt{2}}|v\rangle \,. \tag{15.32}$$

2. The detector does not click (or the bomb remains intact). Although that does not qualify as a measurement in the ordinary sense, it *does* affect the photon's state, an influence that has measurable results. Since the photon was not found in v, it must have gone through u, and the state therefore becomes

$$|\Psi\rangle \xrightarrow{\text{no detection}} \frac{\mathrm{i}}{\sqrt{2}}|u\rangle \,, \tag{15.33}$$

and the interference pattern, at the output of the MZI, breaks:

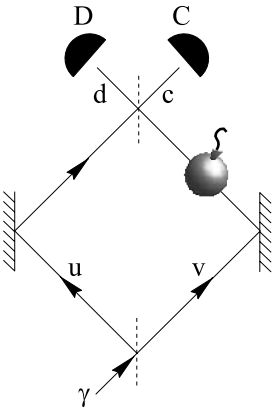

Fig. 15.2. Interaction-free measurement

$$|\Psi\rangle \xrightarrow{\mathrm{BS}_2} \frac{1}{2}(\mathrm{i}|c\rangle - |d\rangle) \,. \tag{15.34}$$

Here, there is a fifty–fifty chance that detector D will click.

This result implies that, by considering only the measurement results from the D detector at the *output* of the interferometer, one can determine whether a measurement was carried out *inside* it, while *no interaction* whatsoever has taken place.

15.5 Hardy's Gedankenexperiment

An interesting twist to the above idea was provided by Hardy [7] (see also Chap. 17). In this case two similar MZI arrangements are considered, one with a positron and one with an electron. When taken separately, the electron can only be detected at its C detector (marked C^-) and the positron at its C detector (marked C^+).

We now consider a situation where the two interferometers are brought together in such a way that the u^- arm of the electron MZI intersects with the u^+ arm of the positron MZI at point A (Fig. 15.3). We assume that, if the electron and positron are both found in this region, then they should annihilate each other with a 100% probability. However, it is possible that each particle will perform an IFM on the other. In that case, there is a 25% probability that *both* particles will hit detector D.

Formally,

$$|\Psi\rangle = |e^-\rangle|e^+\rangle \xrightarrow{\mathrm{BS}_1} \frac{1}{2}(\mathrm{i}|u^-\rangle + |v^-\rangle)(\mathrm{i}|u^+\rangle + |v^+\rangle) \,. \tag{15.35}$$

After the annihilation point A, post-selecting only the cases in which no annihilation took place, the term $|u^-\rangle|u^+\rangle$ is eliminated, leaving us with

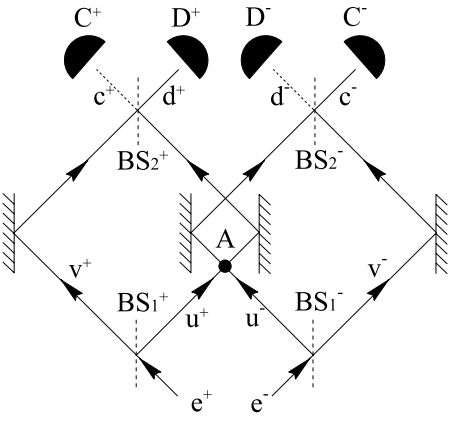

Fig. 15.3. Hardy's paradox

$$|\Psi\rangle \xrightarrow{\text{no annihilation}} \frac{1}{2}\big(\mathrm{i}|u^-\rangle|v^+\rangle + \mathrm{i}|v^-\rangle|u^+\rangle + |v^-\rangle|v^+\rangle\big)\ . \tag{15.36}$$

After passing BS_2,

$$|\Psi\rangle \xrightarrow{\mathrm{BS}_2} \frac{1}{2}\big(\mathrm{i}|c^-\rangle|d^+\rangle + \mathrm{i}|d^-\rangle|c^+\rangle + |d^-\rangle|d^+\rangle\big)\ . \tag{15.37}$$

The interesting case, now, is in the third term, when both D^- and D^+ click. If we try to analyze the situation we conclude as follows:

- The clicking of D^- tells us from the previous example that something disturbed the electron path in the intersecting arm (u^-), hence it could only have gone through the non-intersecting arm (v^-).
- But if this were so, the fact that D^+ clicked tells us that the positron could only have gone through the intersecting arm (u^+).
- But the same logic could be applied starting from the clicking of D^+, in which case we deduce that the positron went through the *non-intersecting* arm (v^+) and the electron through the *intersecting* arm (u^-).
- Clearly, the two alternatives are mutually contradictory.
- A third possibility is then that both the electron and the positron were present in the intersecting arms, but in this case annihilation would have ensued.

Indeed, a paradoxical situation. It seems that it stems from the fact that, in quantum mechanics, one cannot measure which way the particle went without ruining the interference at the output of the MZI. If we try to ascertain which of the above assertions is true, by measuring which way the particles went, we will never get the interference at the output of the interferometers. Weak measurements, however, supply us with a wonderful opportunity to (weakly) obtain which-way information, while keeping the interference intact.

15.6 Weak Measurement Analysis

In order to try and resolve the above paradox, let us recruit the weak measurement. It will allow us to measure the particles as they travel through the interferometer without noticeably affecting their quantum state.

Our main objective is to answer questions like:

- Did the electron take the intersecting arm?
- Where was the positron when the electron went through the non-intersecting arm?

In order to answer questions like the first one, we will use the route occupation operators for the intersecting (I) and non-intersecting (NI) arms:

$$\begin{aligned} \hat{N}(e^+_{\rm NI}) &= |v^+\rangle\langle v^+| \,, \quad & \hat{N}(e^+_{\rm I}) &= |u^+\rangle\langle u^+| \,, \\ \hat{N}(e^-_{\rm NI}) &= |v^-\rangle\langle v^-| \,, \quad & \hat{N}(e^-_{\rm I}) &= |u^-\rangle\langle u^-| \,. \end{aligned} \tag{15.38}$$

In order to answer questions like the second, we will use pair occupation operators, which will measure the simultaneous location of both the electron and the positron:

$$\begin{aligned} \hat{N}(e^+_{\rm NI}, e^-_{\rm I}) &= |v^+\rangle\langle v^+||u^-\rangle\langle u^-| \,, \quad & \hat{N}(e^+_{\rm I}, e^-_{\rm NI}) &= |u^+\rangle\langle u^+||v^-\rangle\langle v^-| \\ \hat{N}(e^+_{\rm I}, e^-_{\rm I}) &= |u^+\rangle\langle u^+||u^-\rangle\langle u^-| \,, \quad & \hat{N}(e^+_{\rm NI}, e^-_{\rm NI}) &= |v^+\rangle\langle v^+||v^-\rangle\langle v^-| \,. \end{aligned} \tag{15.39}$$

One must pay attention to the fact that a weak value of the product of observables is *different* from the product of the weak values of the observables (if $\hat{C} = \hat{A} \cdot \hat{B}$, then $C_{\rm w} \neq A_{\rm w} B_{\rm w}$). Since weak measurements do not interfere with the quantum state, all the above observables can be weakly measured simultaneously:

$$N_{\rm w}(e^+_{\rm NI}) = 0 \,, \qquad N_{\rm w}(e^+_{\rm I}) = 1 \,, \tag{15.40}$$

$$N_{\rm w}(e^-_{\rm NI}) = 0 \,, \qquad N_{\rm w}(e^-_{\rm I}) = 1 \,, \tag{15.41}$$

$$N_{\rm w}(e^+_{\rm I}, e^-_{\rm I}) = 0 \,, \tag{15.42}$$

$$N_{\rm w}(e^+_{\rm NI}, e^-_{\rm I}) = 1 \,, \qquad N_{\rm w}(e^+_{\rm I}, e^-_{\rm NI}) = 1 \,, \tag{15.43}$$

$$N_{\rm w}(e^+_{\rm NI}, e^-_{\rm NI}) = -1 \,. \tag{15.44}$$

Let us now scrutinize these values. According to outcomes (c) and (d) on p. 288, the following counterfactual statement is true:

> If a measurement of some operator $\hat{A}$ were performed between the initiation of the experiment (the pre-selection event) and the measurement of the particles at D^- and D^+ (the post-selection event), while the measurement at the end was still D^- and D^+, then the outcome of the measurement would be equal to the weak value of $\hat{A}$.

This means that the weak values should comply with our basic intuitions with regard to the whereabouts of the electron and the positron. Let us now look at the values and see how our intuition scores.

- $N_{\mathrm{w}}(e^+_{\mathrm{NI}}) = 0$. This means that the positron did not take the non-intersecting arm of the interferometer. That is reasonable, since if it had taken that route, nothing would have obstructed the electron, and the latter would have reached detector C^- instead of D^-.
  It should be remembered that the weak value of an operator has the same value as a counterfactual standard measurement of the same operator that could have been conducted, given that the post-selection condition was still met. If a which-way detector had detected the positron in the non-intersecting route, then the electron could not have met the post-selection condition, that is, reaching detector D^-.
- $N_{\mathrm{w}}(e^+_{\mathrm{I}}) = 1$. The intersecting arm of the positron was occupied. The positron then interfered with the superposition state of the electron and caused it to reach detector D^-.
  Together with the above statement, the fact that the positron was localized in the intersecting arm and not split equally between the two arms means that the superposition state in the interferometer of the positron was damaged, and hence that the positron might reach detector D^+ to meet the post-selection condition.
- $N_{\mathrm{w}}(e^-_{\mathrm{NI}}) = 0$. With the same reasoning as for the non-intersecting arm of the positron, there should be no electron detected in the non-intersecting arm.
- $N_{\mathrm{w}}(e^-_{\mathrm{I}}) = 1$. As a symmetric case to the intersecting arm of the positron, the electron should occupy this arm, otherwise one cannot explain why the positron reached detector D^+. But then, if both the electron and the positron were in the intersecting arms of their interferometers, annihilation must have occurred. That contradicts one of the post-selection conditions of the experiment.
  But wait! We mixed completely different questions here. Remember that the weak value of the product of two operators is not equal to the product of the weak values. The questions as to whether the electron occupied the intersecting arm or the positron occupied the intersecting arm are different from the question: Did both the electron and the positron occupy their respective intersecting arms at the same time? The answer to the latter is the pair-occupation value below.
- $N_{\mathrm{w}}(e^+_{\mathrm{I}}, e^-_{\mathrm{I}}) = 0$. Here the weak value satisfies our expectations – both the electron and the positron did not occupy the intersecting arms at the same

time. One must keep in mind that the weak values represent counterfactual measurements. Although the weak values can be measured together, one cannot conduct the experiments that measure these ('strong') values simultaneously. That is why the two single-particle occupation numbers do not comply with the pair occupation value.

- $N_{\mathrm{w}}(e^{+}_{\mathrm{NI}}, e^{-}_{\mathrm{I}}) = 1$. There is a particle pair in the non-intersecting arm of the positron and the intersecting arm of the electron. This is because the electron 'collapsed' to the intersecting arm, and forced the positron to collapse to the other arm. In this configuration, no annihilation will occur and the two particles will be detected in their D detectors – as required by the post-selection condition.
- $N_{\mathrm{w}}(e^{+}_{\mathrm{I}}, e^{-}_{\mathrm{NI}}) = 1$. The symmetric measurement is also true. If one tries to look for the positron in the intersecting arm and the electron in the non-intersecting arm, they will be found there with certainty.
  Note the peculiar effect of post-selection: it does not matter whether one (strongly) measures the intersecting arm of the electron and the non-intersecting arm of the positron, or vice versa. If the post-selection condition is met at the end, then either measurement will find the pair with probability 1!
  Now, if one recalls outcome (a) on p. 288, the weak value of the sum of two operators is the sum of their weak values. Summing the weak values of the pair occupation numbers gives the paradoxical result that *there are 2 pairs of particles in the interferometers!*
  Luckily, the mathematics converges with the fourth pair occupation number below.
- $N_{\mathrm{w}}(e^{+}_{\mathrm{NI}}, e^{-}_{\mathrm{NI}}) = -1$. The pair occupation number of the two non-intersecting arms is -1. What does that mean?
  First of all, it settles the counting problem for the number of pairs in the interferometers. The sum of all four possible pairs is now one, which is consistent with the fact that there is only one pair of particles in the interferometer.
  But still, what does it mean to have a *minus-one* pair of particles in the non-intersecting arms? Bear in mind that a measurement of pair occupation number is different from two measurements, one for each of the particles. The measurement must look for the presence of the pair without being able to tell the presence of each particle. One such measurement could be putting the two particles in sealed boxes, then bringing the two boxes close enough to each other, and letting the electrostatic force pull the boxes even closer together.
  In such an experiment, a minus-one pair of particles will manifest itself by the *repulsion* of the boxes, instead of the boxes being attracted in the usual case (Fig. 15.4). Of course, since the measurement is weak, the pair of boxes will move only slightly – and that movement will not be enough

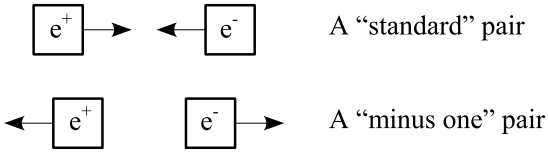

Fig. 15.4. Weak measurement of a ‘minus-one’ pair of particles

to conclude a firm result. But repeating the experiment with many such pairs will give an average movement away from each other.

Resch et al. [8] performed a quantum optics experiment to realize another weak measurement problem. In their setup, a weak measurement was supposed to find a ‘minus-one particle’. They used a tilted glass flat to weakly measure the existence of a particle: a normal particle, traversing the flat, should displace the beam slightly (as in Fig 15.5). In order to achieve a weak measurement, the displacement must be substantially smaller than the width of the beam. In their experiment, the existence of a minus-one particle was manifested by a *negative* displacement of the beam. The group is now working on a realization of Hardy’s paradox with weak measurement using quantum optics.

One should remember that there is no electron in the non-intersecting arm $[N_{\mathrm{w}}(e^{-}_{\mathrm{NI}}) = 0]$, but still, there is one electron–positron pair where the positron is in the intersecting arm and the electron is in the non-intersecting arm $[N_{\mathrm{w}}(e^{+}_{\mathrm{I}}, e^{-}_{\mathrm{NI}}) = 1]$. How can these two statements be valid given the weak value summation rule?

The answer lies in the minus-one pair in the non-intersecting arms. When this pair is taken into account, it cancels the plus-one pair mentioned above, bringing the total number of electrons in the non-intersecting arm to zero:

$$N_{\mathrm{w}}(e^{-}_{\mathrm{NI}}) = N_{\mathrm{w}}(e^{+}_{\mathrm{I}}, e^{-}_{\mathrm{NI}}) + N_{\mathrm{w}}(e^{+}_{\mathrm{NI}}, e^{-}_{\mathrm{NI}}) = 1 - 1 = 0\,. \tag{15.45}$$

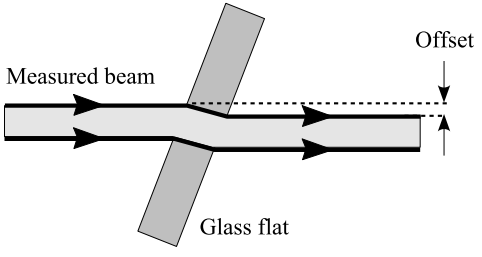

Fig. 15.5. An optical weak measurement of a particle

15.7 Conclusion

In this chapter the concepts of weak measurement and weak values were used to resolve the paradox raised by Hardy. As opposed to the orthodox analysis of the conundrum, which reached a dead end and labeled some questions as 'unaskable', the weak values allow us to give proper answers to these questions, along with the new possibilities of simultaneously measuring non-commuting variables, simultaneously measuring the results of counterfactual experiments, and revealing a negative number of particles.

The concepts of weak measurement and weak values arise from the two-vector formalism. The two-vector is a strict interpretation of quantum mechanics. As such, it cannot lead to experimental results that depart from the orthodox quantum analysis. Nonetheless, it has proved to be a source of ingenious ideas, paradoxes and insights.

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16 Non-Commutative Quantum Geometry: A Reappraisal of the Bohm Approach to Quantum Theory

B.J. Hiley

In this paper we trace the mathematical origins of the wave and particle aspects of quantum phenomena to the symplectic symmetry, $Sp(2n)$. This symmetry is shared by both classical mechanics and quantum mechanics. We show how the quantum formalism appears in the covering space of the symplectic group. In the quadratic Hamiltonian approximation, it is the covering metaplectic group, $Mp(2n)$, that gives the Schrödinger equation directly. As is well known, it is the generalisation to all Hamiltonians, to $Ham(2n)$, that presents mathematical difficulties. Recently, de Gosson has shown how to derive the Schrödinger equation even in this case. Our approach is to address the problem through the non-commutative algebraic approach, which has its origin in the work of Emch. We show how the Schrödinger equation and its dual can be expressed in a totally algebraic form that involves both the commutator and the anticommutator (or Jordon product). We show that the resulting two equations project directly into the equations forming the basis of the Bohm interpretation. In fact we can regard the Bohm approach as providing a way to construct shadow manifolds expected from the ideas of non-commutative geometry. Finally we discuss some of the consequences that follow from this mathematical structure.

16.1 Introduction

I want to use this opportunity to review the new developments that have taken place since the appearance of the book *The Undivided Universe* which I completed with David Bohm just before he died (Bohm and Hiley 1987, 1993). In some of the literature, this approach has been known as the de Broglie–Bohm approach or, more recently, as Bohmian mechanics. But it should be noted that discussions using these names sometimes differ in important respects from what Bohm and I had in mind when we wrote our book. I do not want to go into these differences here as I would prefer to go directly to the new developments that have emerged from a fusion of my own ideas (Brown and Hiley 2000, Hiley 2001, 2002 & 2003) with those of de Gosson (2001). This work throws a very different light on the formalism first proposed by Bohm (1952). In this paper I would like to summarise the main

developments that have occurred and to explain how I now see the Bohm approach.

I want to start by recalling some of the key steps in Schrödinger's original attempt to derive his equation from classical physics. Of course he failed to provide a mathematically 'clean' derivation, as he himself acknowledged in his original paper. However, it has been realised for some time now that it is possible to derive the Schrödinger equation rigorously from classical symplectomorphisms by lifting the classical phase space behaviour onto a covering space, provided the Hamiltonian is up to quadratic in position and momentum (see Guillemin and Sternberg 1990 and de Gosson 2001). It this lifting process that enables the wave and particle aspects to be described in a united formalism. In effect, the particle properties are described on the underlying phase space while the wave properties appear at the level of the covering space. A summary of these ideas will be discussed in Sect. 16.2.

The lift onto a more generalised covering space, $Ham(2n)$, has presented considerable mathematical difficulties (see Guillemin and Sternberg 1990). De Gosson (2001) has shown recently how these difficulties can be avoided and the Schrödinger equation can be derived rigorously for all Hamiltonians. In Sect. 16.3, I will describe how the mathematical difficulties of $Ham(2n)$ can also be avoided by using some of the ideas that are available from non-commutative geometry. This has meant applying the purely algebraic approach to quantum mechanics using the ideas detailed in Emch (1972). In more modern terms, this exploits the properties of the symplectic Clifford algebra and uses the less well-known structure of the symplectic spinor (see Crumeyrolle 1990). Because this algebraic structure is non-commutative it is not possible to obtain a unique underlying phase space, as one can from a commutative structure. What one is forced to do is to construct so-called shadow manifolds and one such manifold is the Bohm phase space. Here we will see the precise reason for the appearance of the quantum potential, which some physicists regard as ad hoc. It turns out that its appearance is a direct consequence of projecting the non-commutative algebraic structure onto a shadow manifold.

In Sect. 16.4, we show that (infinitely) many shadow phase spaces can be constructed, each with their own quantum potential. Thus it is possible to obtain a Bohm approach in the momentum representation, so removing the criticism that the original proposals of Bohm produced an asymmetry that is not in the original formalism. Our work shows that the Bohm approach is deeply embedded in the standard formalism and simply provides an alternative perspective to it. What Bohm does is simply choose the position representation to be special and develops the interpretation on that basis. I believe our work shows that there are no scientific grounds for arguing that the Bohm interpretation is in some way fundamentally flawed. It is simply another way of looking at the same formalism. We conclude by discussing the consequences of this new way of looking at the Bohm interpretation.

16.2 The Schrödinger Equation

To obtain his equation, Schrödinger (1926) started with the time independent classical Hamiltonian–Jacobi equation

$$H(\boldsymbol{r}, \nabla S) = E \,, \tag{16.1}$$

where S is the classical action. Schrödinger then wrote $S = K \ln \psi$ so that the Hamilton–Jacobi equation for a particle in a classical potential V becomes

$$(\nabla\psi)^2 - \frac{2m}{K}(E - V)\psi^2 = 0 \,. \tag{16.2}$$

To obtain the wave equation from (16.2), Schrödinger assumed that H was a quadratic function in ψ and its derivatives. He then performed the variation

$$\delta J = \delta \int\int\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \left[(\nabla\psi)^2 - \frac{2m}{K}(E + V)\psi^2\right] = 0 \,.$$

This immediately gives the Schrödinger equation for a particle with constant energy in a classical potential V. Of course, it was not clear even to Schrödinger what this variation means because he writes in a footnote: "I realise that this formulation in not unambiguous" (Schrödinger 1926).

Thus we see that historically the Hamilton–Jacobi equation played an important role in leading to the Schrödinger equation. The Hamilton–Jacobi equation also plays a key role in the Bohm interpretation, which we shall discuss in Sect. 16.3. The main question that I want to raise here is whether we can derive the Schrödinger equation from the classical formalism in a rigorous way. I should say that this work was partially motivated by Polkinghorne's remarks that this equation "came out of Schrödinger's head" (Polkinghorne 2002).

16.2.1 Symplectic Symmetries of Classical Mechanics

To show how we can derive the Schrödinger equation from classical mechanics let us first recall how the symplectic group arises in classical mechanics. We begin with Hamilton's equations of motion, which can be written succinctly in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(x, p) = X_H(x, p) \,, \tag{16.3}$$

where $X_H = (\nabla_p H, -\nabla_x H)$. By solving these equations we can write the dynamics in the form

$$(x, p) = f_{t,t_0}(x_0, p_0) \,, \tag{16.4}$$

where f_{t,t_0} is a symplectomorphism. Infinitesimal symplectomorphisms s_{t,t_0} satisfy the relation

$$\tilde{s}Js = J \ , \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ .$$

In practice these can be most easily found, not by solving Hamilton's equation of motion, but by solving the corresponding Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H(\boldsymbol{r}, \nabla S) = 0 \ ,$$

which for a particle in a classical potential $V(\boldsymbol{r}, t)$ takes the form

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V = 0 \ . \tag{16.5}$$

We now illustrate how this works for a free particle in one dimension. We first solve the Hamilton–Jacobi equation and find

$$S(x, x_0, t, t_0) = \frac{m(x - x_0)^2}{2(t - t_0)} \ , \tag{16.6}$$

with $p = \partial S/\partial x$ and $p_0 = -\partial S/\partial x_0$. It is then easy to show that

$$x = x_0 + \left(\frac{t - t_0}{m}\right) p_0 \ , \quad p = p_0 \ , \quad f_{t,t_0} = \begin{pmatrix} 1 & \dfrac{t - t_0}{m} \\ 0 & 1 \end{pmatrix} \ . \tag{16.7}$$

We can also easily verify that f_{t,t_0} is a symplectic matrix.

We can carry through the same procedure for the one-dimensional harmonic oscillator using the function

$$S_{\mathrm{HO}} = \frac{m\omega}{2 \sin \omega t} \left[\left(x^2 + x_0^2\right) \cos \omega t - 2 x x_0 \right] \ , \tag{16.8}$$

to obtain the well-known equation of motion. In this way we see how $S(\boldsymbol{r}, \boldsymbol{r}_0, t, t_0)$ generates the classical motion.

16.2.2 Ray Optics

In order to see why the symplectic group plays a key role in the derivation of a wave equation from particle mechanics, let me briefly recall how geometric optics is also based on this group structure. Recall Fermat's principle in which the equation of a light ray can be derived from the variation of the optical path

$$\int n \mathrm{d}\sigma = \int n \sqrt{1 + \dot{x}^2 + \dot{y}^2} \, \mathrm{d}z = \int L \, \mathrm{d}z \ , \tag{16.9}$$

where we have written

$$\mathrm{d}\sigma = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2} = \left[1 + \left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2\right]^{1/2} = \sqrt{1 + \dot{x}^2 + \dot{y}^2}\,\mathrm{d}z\ ,$$

so that

$$L = \sqrt{1 + \dot{x}^2 + \dot{y}^2}\ .$$

Obtaining the conjugate momenta $p = \mathrm{d}L/\mathrm{d}z$ from L, we find

$$p_x = n\frac{\dot{x}}{\sqrt{1 + \dot{x}^2 + \dot{y}^2}}\ , \quad p_y = n\frac{\dot{y}}{\sqrt{1 + \dot{x}^2 + \dot{y}^2}}\ ,$$

so that the equivalent Hamiltonian is

$$H = p_x\dot{x} + p_y\dot{y} - L = -\sqrt{n^2 - p_x^2 - p_y^2}\ . \tag{16.10}$$

Hamilton's equations of motion are then

$$\dot{x} = \frac{\partial H}{\partial p_x}\ , \quad \dot{p}_x = -\frac{\partial H}{\partial x}\ , \quad \dot{y} = \frac{\partial H}{\partial p_y}\ , \quad \dot{p}_y = -\frac{\partial H}{\partial y}\ .$$

The equation corresponding to the Hamilton–Jacobi equation (16.1) is then

$$\frac{\partial S}{\partial z} - \sqrt{n^2 - \left(\frac{\partial S}{\partial x}\right)^2 - \left(\frac{\partial S}{\partial y}\right)^2} = 0\ , \quad (\nabla S)^2 - n^2 = 0\ , \tag{16.11}$$

which is just the well-known eikonal equation.

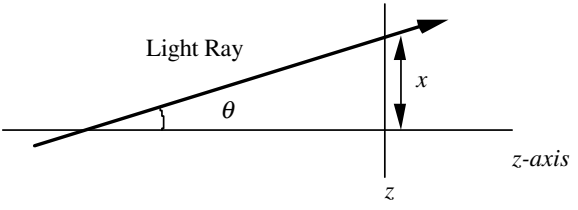

Fig. 16.1. Coordinates $(x, p = n\theta)$ of light ray

To identify the coordinates (x, p_x), consider a ray that travels in the x–z plane, where x is the height of the ray above the x-axis at the point z. Hence, $p = n\theta$, where θ is the angle the ray makes with the z-axis (see Fig. 16.1).

In terms of these coordinates, the equation of the ray can also be written as a symplectomorphism, which we can write in the form

$$\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_0 \\ p_0 \end{pmatrix}\ . \tag{16.12}$$

Then, for a ray travelling in uniform medium of refractive index n, we have

$$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} , \quad d = \frac{\Delta t}{n} , \tag{16.13}$$

while for a ray passing through a boundary $n_1 \to n_2$, we have

$$\begin{pmatrix} 1 & 0 \\ -P & 1 \end{pmatrix} , \quad P = \frac{n_1 - n_2}{R} .$$

Then for a lens, the symplectomorphism is given by

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - u/f & u + v - uv/f \\ -1/f & 1 - v/f \end{pmatrix} .$$

If the planes are conjugate, the upper off-diagonal element is zero. From this we can find the thin lens formula

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f} .$$

I derive it in this manner just to stress that underlying ray optics and classical mechanics is the symplectic group. I now want to make use of this fact to motivate the notion of a covering space, which we will use to describe the wave properties generated by both types of process.

16.2.3 Waves and Rays from Huygens's Construction

Let us now recall how a series of wave fronts is related to the eikonal equation. Consider a wave front given by the equation $ct = S(x, y, z)$, so that

$$c\mathrm{d}t = \mathrm{d}S = \frac{\partial S}{\partial x}\mathrm{d}x + \frac{\partial S}{\partial y}\mathrm{d}y + \frac{\partial S}{\partial z}\mathrm{d}z . \tag{16.14}$$

Let the ray P_0P perpendicular to the wave fronts be described by direction cosines $\alpha = \partial x/\partial\sigma$, $\beta = \partial y/\partial\sigma$, and $\gamma = \partial z/\partial\sigma$, with $\alpha^2 + \beta^2 + \gamma^2 = 1$, so that

$$N\alpha = \frac{\partial S}{\partial x} , \quad N\beta = \frac{\partial S}{\partial y} , \quad N\gamma = \frac{\partial S}{\partial z} .$$

Substituting into (16.14), we find

$$\begin{aligned} c\mathrm{d}t &= N(\alpha\,\mathrm{d}x + \beta\,\mathrm{d}y + \gamma\,\mathrm{d}z) \\ &= N\left[\left(\frac{\partial x}{\partial\sigma}\right)^2 + \left(\frac{\partial y}{\partial\sigma}\right)^2 + \left(\frac{\partial z}{\partial\sigma}\right)^2\right]\mathrm{d}\sigma = N\,\mathrm{d}\sigma . \end{aligned}$$

Now if the light has velocity v in a refractive medium, then $\mathrm{d}\sigma = v\mathrm{d}t$ and $n = c/v$, so that $c\mathrm{d}t = n\mathrm{d}\sigma$ and $N = n$. Since $\alpha^2 + \beta^2 + \gamma^2 = 1$, we have

$$\frac{1}{n^2}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{n^2}\left(\frac{\partial S}{\partial y}\right)^2 + \frac{1}{n^2}\left(\frac{\partial S}{\partial z}\right)^2 = 1 ,$$

which is just the eikonal equation (16.11).

16.2.4 The Wave Equation

In the optical case considered in this section, we begin to see how rays and waves are related. What remains to be done is to describe how the phase changes with time. To begin with, note that the solution of the eikonal equation is

$$S(P_0, P) = \frac{n}{2} \frac{(x - x_0)^2}{t - t_0} \,, \tag{16.15}$$

which should be compared with the solution of the Hamilton–Jacobi equation for the free particle given in (16.6). Here this is simply the optical path length between P_0 and P. This will produce the phase change $\exp\big[2\pi \mathrm{i} k S(P_0, P)\big]$, where $k = 1/\lambda$, λ being the wavelength of the light. As the light travels along its path, the amplitude decreases by some factor depending on the length of the path, so that the wave will be modulated by

$$A(P_0, P) \exp\big[2\pi \mathrm{i} k S(P_0, P)\big] \,.$$

If initially we have the wave $\psi_0(x_0)$, then the contribution to the wave at x from the secondary source at x_0 will be

$$\psi(x) = A(x, x_0) \exp\big[2\pi \mathrm{i} k S(x, x_0)\big] \psi_0(x_0) \,.$$

If we now assume that the Huygens construction is valid, we get

$$\psi(x) = \int A(x, x_0) \exp\big[2\pi \mathrm{i} k S(x, x_0)\big] \psi_0(x_0) \, \mathrm{d}x_0 \,. \tag{16.16}$$

[Note that the analysis has been shortened here. In fact, there is a slight complication, which is fully discussed in Guillemin and Sternberg (1990). This does not affect our conclusions.]

Feynman (1948) shows that (16.16) is exact for free particles provided we use $S(x, x_0)$ as defined in (16.6). Does this mean that we are returning to the Feynman path integral approach? Not according to de Gosson (2001)! He argues that, provided we accept the de Broglie matter wave hypothesis, the Schrödinger equation emerges from classical mechanics not only for quadratic Hamiltonians, but for all Hamiltonians. Furthermore, the emergence is mathematically exact. To explain his argument here would take us too far from the main point I am making in this paper, so I will refer the interested reader to his excellent book. For the purposes of this paper, the point I want to bring out is the connection of this work to the Bohm interpretation.

16.3 Covering Groups

16.3.1 The Symplectic and Metaplectic Groups

From the mathematical point of view, what the Huygens construction has done is to lift the Hamiltonian flow from the classical phase space onto the

covering space. For quadratic Hamiltons this generates the metaplectic group, which is the double cover of the symplectic group. (For more details see Guillemin and Sternberg 1978, 1990 and de Gosson 2001).

For the purposes of this paper let us define an operator U_S by

$$(U_S\psi)(x) = A\int \exp\left[2\pi \mathrm{i}kS(x,x_0)\right]\psi_0(x_0)\,\mathrm{d}x_0\ . \tag{16.17}$$

Then obviously,

$$\left[(U_{S_1}\circ U_{S_2})\psi\right](x) = A\int \exp\left\{2\pi \mathrm{i}k\left[S_1(x,x_0)+S_2(x,x_0)\right]\right\}\psi_0(x_0)\,\mathrm{d}x_0\ ,$$

so that

$$U_{S_1}\circ U_{S_2} = U_{S_1+S_2}\ . \tag{16.18}$$

Thus the covering clearly has a group structure. If we take the special case where $S(x,x_0)$ is defined by

$$S(x,x_0) = \frac{1}{2B}\left(Dx^2 - 2xx_0 + Ax_0^2\right)\ , \tag{16.19}$$

we find that that this generates the metaplectic group, $Mp(2n)$, which double covers the symplectic group generated by

$$s = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\ ,$$

in $2n$-dimensional phase space. The matrices A, B, C and D satisfy the relations:

- $\tilde{A}C$ and $\tilde{D}B$ are symmetric, $\tilde{A}D - \tilde{C}D = 1$,
- $A\tilde{B}$ and $C\tilde{D}$ are symmetric, $A\tilde{D} - B\tilde{C} = 1$,
- $D\tilde{C}$ and $A\tilde{B}$ are symmetric, $D\tilde{A} - C\tilde{B} = 1$,

where the tilde signifies the transpose of the matrix.

16.3.2 Schrödinger Equation

One can find in the metaplectic group families of operators $U(t) \in Mp(2n)$ that form a one-parameter subgroup satisfying $U(t_1)U(t_2) = U(t_1 + t_2)$. Corresponding to this one-parameter group there will be a one-parameter group of matrices $M(t) \in Sp(2n)$. Since M depends on t, we can always write $\mathrm{d}M(0)/\mathrm{d}t = K$, so that $M(t) = \mathrm{e}^{tK}$. Now for small $|t|$, we can uniquely recover $U(t)$ from $M(t)$. Once we know $U(t)$ for small $|t|$, we know it for all t because $U(t) = U(t/n)^n$. Since $\mathrm{d}U/\mathrm{d}t$ is a skew adjoint operator, we can write $\mathrm{d}U/\mathrm{d}t = -\mathrm{i}H$, where H is a self-adjoint operator. Thus $U(t)$ satisfies the Schrödinger-type equation

$$\frac{\mathrm{d}U}{\mathrm{d}t} = -\mathrm{i}HU \ . \tag{16.20}$$

By examining the details of the matrix f_{t,t_0} given in (16.7), it is not difficult to show that t is time. This approach is exact for Hamiltonians that are up to quadratic in x and p. This is clearly shown in (16.19).

Before going on to explain how to generalise this approach to all Hamiltonians, let us just display the connection more explicitly for the free particle and the one-dimensional harmonic oscillator. In classical mechanics, we have the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H(\boldsymbol{r}, \nabla S) = 0 \ .$$

The solution $S(\boldsymbol{r}, \boldsymbol{r}_0, t, t_0)$ of this equation generates the classical motion. Thus, for the free particle, we find

$$S_{\mathrm{FP}} = \frac{m(\boldsymbol{r} - \boldsymbol{r}_0)^2}{t - t_0} \ , \tag{16.21}$$

while the generating function for the harmonic oscillator is

$$S_{\mathrm{HO}} = \frac{m\omega}{2\sin\omega t}\left[\left(x^2 + x_0^2\right)\cos\omega t - 2xx_0\right] \ . \tag{16.22}$$

Both of these are special cases of the quadratic expression shown in (16.19).

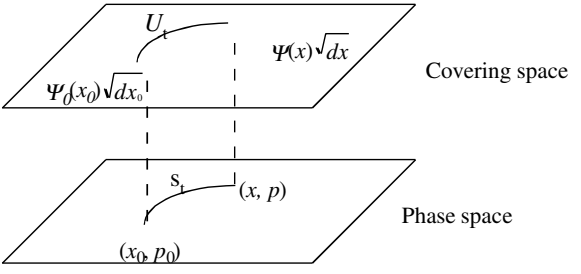

Fig. 16.2. The relation between the classical phase space and the covering space of quantum mechanics

Now let us compare these equations with the solutions of the Schrödinger equation (16.20), which we now write in the more familiar form

$$\mathrm{i}\frac{\partial \psi}{\partial t} = H\psi \ .$$

The Green's function solution, $G(\boldsymbol{r}, \boldsymbol{r}_0, t, t_0)$, of Schrödinger's equation propagates the quantum motion. For the free particle, the Green's function is

$$G_{\rm FP} \propto \exp\left[\mathrm{i}\frac{m(\boldsymbol{r}-\boldsymbol{r}_0)^2}{t-t_0}\right] = \exp\left[\mathrm{i}S_{\rm FP}(\boldsymbol{r},\boldsymbol{r}_0)\right] , \tag{16.23}$$

while the harmonic oscillator gives

$$G_{\rm HO} \propto \exp\left\{\frac{\mathrm{i}m\omega}{2\sin\omega t}\left[(x^2-x_0^2)\cos\omega t - 2xx_0\right]\right\} = \exp\left[\mathrm{i}S_{\rm HO}(\boldsymbol{r},\boldsymbol{r}_0)\right] . \tag{16.24}$$

Note here that the Green's functions are just the lift of the classical Hamiltonian flow and are clearly directly related to the representation of the operators in the metaplectic group $Mp(2n)$. Figure 16.2 shows an image of how the classical phase space is covered by the space in which the metaplectic group operates.

16.3.3 Symplectic Spinors

The object that takes the place of the coordinates (x,p) of classical phase space is the symplectic spinor $\psi(x)\sqrt{\mathrm{d}x}$. This is also known as a half-form. It is simply the symplectic analogue of the Pauli spinor. The root of $\mathrm{d}x$ appears because of the normalisation condition that holds both in electromagnetism and in quantum mechanics. That is, we must have

$$\int |\psi(x)|^2\mathrm{d}x = \int |\psi_0(x_0)|^2\mathrm{d}x_0 . \tag{16.25}$$

Thus (16.16) should be written in the form

$$\psi(x)\sqrt{\mathrm{d}x} = A\int \exp\left[2\pi\mathrm{i}kS(x,x_0)\right]\psi_0(x_0)\sqrt{\mathrm{d}x_0} . \tag{16.26}$$

If we substitute (16.26) into the normalisation condition (16.25), we find that $A = \mathrm{e}^{\mp\mathrm{i}\pi/4}|\det\lambda B|^{-1/2}$. With this result, we can now show how the double cover arises.

To this end, first notice that, when $A=0$, $B=1$, $C=-1$, $D=0$, the transformation becomes the Fourier transform $(x,p) = F(x_0,p_0)$. This corresponds to a rotation through $\pi/2$ in the phase space. Thus

$$\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ p_0 \end{pmatrix} ,$$

so that $x_0 \to -p$ and $p_0 \to x$. Clearly we need to apply the Fourier transformation four times to return to our starting point, i.e., four lots of $\pi/2$ gives 2π! Thus since $F(Ff)(x) = f(-x)$, we get $F^4 = 1$.

Now let us see what happens when we go to the double cover. The metaplectic group element corresponding to the Fourier transformation on the phase space is

$$U_F\psi(x) = \mathrm{e}^{-\pi\mathrm{i}/4}|k|^{1/2}\int \exp\big[-2\pi\mathrm{i}kxx_0\big]\psi_0(x_0)\,\mathrm{d}x_0\ , \tag{16.27}$$

so that $U_F = \mathrm{e}^{-\pi\mathrm{i}/4}F$. This gives immediately

$$(U_F)^4 = \Big(\mathrm{e}^{-\pi\mathrm{i}/4}F\Big)^4 = -F\ . \tag{16.28}$$

This means that we have to go through another four operations with the group element (16.26) to get back to where we started. This is a well-known property of any double cover, which has physical consequences demonstrated by the Guoy effect (see Carpenter 1959).

16.3.4 Heisenberg Algebra and the Metaplectic Group

In this section, I will show how these ideas are contained in the Heisenberg group and its corresponding Lie algebra, which can also provide an alternative and perhaps more formal way of showing the equivalence of the Schrödinger and Heisenberg pictures which we will exploit later.

Recall that the Heisenberg algebra is generated by $\{1, Q, P\}$, with

$$[Q, P] = \mathrm{i}\ .$$

It is not difficult to show that the enveloping algebra of the Heisenberg algebra is part of a larger structure known in the mathematics literature as the symplectic Clifford algebra (see Bacry and Boon 1987, and Crumeyrolle 1990). The reason for this name is that it is the symplectic analogue of the more familiar orthogonal Clifford algebra, which is central to the Dirac theory of the electron. Just as there is a double cover of the orthogonal group which gives rise to the well known Pauli and Dirac spinors, the symplectic group has a double cover known as the metaplectic group which in turn gives rise to symplectic spinors discussed in the previous section. It is these symplectic spinors that play a central role in this paper.

The corresponding group, the Heisenberg group, is spanned by the elements

$$H(\alpha, \beta, \varepsilon) = \exp(\alpha Q + \beta P + \varepsilon Z)\ , \quad \text{for } Z \in \mathbb{R}\ , \tag{16.29}$$

while the metaplectic group (the corresponding Clifford group of the symplectic structure) is spanned by the elements

$$M(\alpha, \beta, \varepsilon) = \exp\big[\alpha Q^2 + \beta P^2 + \varepsilon(QP + PQ)\big]\ . \tag{16.30}$$

The infinitesimal generators of this group are

$$L_0 = \frac{1}{4}\big[Q^2 + P^2\big]\ , \quad L_1 = -\frac{1}{4}\big[QP + PQ\big]\ , \quad L_2 = -\frac{1}{4}\big[Q^2 - P^2\big]\ ,$$

producing a Lie algebra

$$\big[L_0, L_1\big] = \mathrm{i}L_2 \ , \quad \big[L_1, L_2\big] = -\mathrm{i}L_0 \ , \quad \big[L_2, L_0\big] = \mathrm{i}L_1 \ .$$

Clearly, this is isomorphic to the symplectic Lie algebra $Sp(4)$. As I have shown elsewhere, we can extend the symplectic Clifford algebra so that it contains primitive idempotents (Hiley 2001a). Then it is possible to construct the minimal ideals which are the algebraic equivalent of the symplectic spinors of the metaplectic group, as we will show in Sect. 16.5. We will also discuss the relevance of the algebraic spinors in Sect. 16.5.

16.4 Quantum Mechanics and the Bohm Approach

16.4.1 Summary of the Bohm Approach

We now want to consider what all this has to do with the Bohm (1952) approach. (For a more up to date account of this approach see Bohm and Hiley 1993 and Holland 1993). First recall that the two equations defining the Bohm approach emerge from the Schrödinger equation by simply writing the wave function in polar form: $\psi(\boldsymbol{r}, t) = R(\boldsymbol{r}, t) \exp\big[\mathrm{i}S(\boldsymbol{r}, t)\big]$. Then the resulting equation is split into its real and imaginary parts and we find

$$\mathrm{i}\frac{\partial P}{\partial t} + \nabla(jP) = 0 \ , \tag{16.31}$$

which gives us a conservation of probability equation. The real part of the Schrödinger equation gives

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} - \frac{1}{2m}\frac{\nabla^2 R}{R} + V = 0 \ . \tag{16.32}$$

This equation resembles the Hamilton–Jacobi equation except that it contains an extra term

$$Q = \frac{1}{2m}\frac{\nabla^2 R}{R} \ ,$$

which has been called the quantum potential since it is this term that distinguishes classical mechanics from quantum mechanics. If we identify ∇S with the momentum and regard Q as a new quality of energy only playing a role in quantum processes, then we can regard (16.32) as an expression of the conservation of energy.

It is this term that has provoked some hostile criticism to the Bohm approach. Heisenberg (1958) himself called it ad hoc, a sentiment that is still repeated in Polkinghorne (2002). Yet there is nothing ad hoc about it. It is a direct consequence of simply rewriting the Schrödinger equation in its real and imaginary parts under polar decomposition of the wave function. Even

Dürr, Goldstein and Zanghi (1996) who are strong supporters of 'Bohmian mechanics' seem to find "a serious flaw in the quantum potential approach," but I have failed to understand their specific objections, particularly as they argue that the approach is "completely defined by the Schrödinger equation."

I find the fact that (16.32) emerges from the Schrödinger equation far from surprising given the history of the origin of this equation. Indeed we can see even more clearly how the quantum potential is a necessary feature of the description if we follow the suggestion of de Gosson (1998, 2001) and introduce what he calls the Bohmian defined by

$$H^{\psi} = H + Q^{\psi} \ .$$

If this is inserted into the Hamilton–Jacobi equation written in the form

$$\frac{\partial S}{\partial t} + H^{\psi}(\boldsymbol{r}, \nabla_{\boldsymbol{r}} S) = 0 \ , \tag{16.33}$$

then one can show that there exists a symplectomorphism f^{ψ}_{t,t_0} given by $\big(\boldsymbol{r}^{\psi}(t), \boldsymbol{p}^{\psi}(t)\big) = f^{\psi}_{t,t_0}(\boldsymbol{r}_0, \boldsymbol{p}_0)$, which can be written in the form

$$\frac{\mathrm{d}\boldsymbol{r}^{\psi}}{\mathrm{d}t} = \nabla_{\boldsymbol{p}} H^{\psi} \tag{16.34}$$

and

$$\frac{\mathrm{d}\boldsymbol{p}^{\psi}}{\mathrm{d}t} = -\nabla_{\boldsymbol{r}} H^{\psi} = -\nabla_{\boldsymbol{r}}(V + Q^{\psi}) \ . \tag{16.35}$$

Equation (16.34) is simply the guidance condition $\boldsymbol{p} = \nabla S$, from which the trajectories are calculated, written in an unusual form. Equation (16.35) is the generalisation of Newton's equation of motion where the classical potential is supplemented with the quantum potential. This equation ensures that the momentum is always conserved.

If we write

$$S\big(\boldsymbol{r}^{\psi}(t), t\big) = S_0(x_0) + \int_0^t (\boldsymbol{p}\cdot \mathrm{d}\boldsymbol{r} - H^{\psi}\mathrm{d}t') \ ,$$

then it is not difficult to show that

$$\psi\big(\boldsymbol{r}^{\psi}(t), t\big)|\mathrm{d}^n \boldsymbol{r}^{\psi}|^{1/2} = \exp\left[\frac{\mathrm{i}}{\hbar} S\big(\boldsymbol{r}^{\psi}(t), t\big)\right] \psi_0(x_0)|\mathrm{d}^n x_0|^{1/2} \ , \tag{16.36}$$

where $\psi\big(\boldsymbol{r}^{\psi}(t), t\big)$ is a solution of Schrödinger's equation. For proof of these results, see de Gosson (1998). This demonstrates the key role the quantum potential plays in the relation between the symplectic space and its double cover, as we will elaborate further below.

Now we are in a position to clarify the relation between the phase space and its covering space. We can regard the Schrödinger equation as describing

the time evolution of the flow in the quantum system in the covering space. Then, for each initial point of the covering space, we can project a distribution of initial points on the phase space lying immediately below it. As the Schrödinger equation develops, a series of trajectories unfold in the underlying manifold. These determine classical-type symplectic flows, which are defined by the Bohmian. Thus we see that the quantum potential plays an essential role in the mathematical relationship between the phase space and the corresponding covering space. Furthermore, it is the property of covering spaces that ensures that the underlying trajectories do not cross, thus explaining a well-known property of the Bohm trajectories. There is therefore nothing ad hoc or artificial about the Bohm flows. They are a necessary structure of the quantum formalism when looked at in terms of the underlying geometry. We will look at this result in a new way in the next section.

Before moving on, I would like to consider several features of the Bohm approach that are sometimes considered to be 'unsatisfactory'. Firstly, the Bohm approach seems to depend heavily on the Schrödinger picture and totally ignores the Heisenberg picture. Yet the claim is that the Bohm approach uses only the standard formalism. Where then are things like the quantum potential 'hiding' in the Heisenberg picture? It must be there somewhere simply because the matrix mechanics gives a mathematically equivalent description, at least for a finite number of degrees of freedom. We will examine this question in the next section.

A further question, which essentially has it roots in the first question, asks why the position representation is taken as basic. It seems as if it completely ignores the complementary momentum representation. Bohm (1953) did address this question and concluded that one could not build a satisfactory momentum version because of problems with the mathematics. However, this is not so, as we will show later.

The third difficulty is to do with the uncertainty principle. How is it possible to build a phase space in terms of (x, p) and yet satisfy the uncertainty principle and avoid the no-go theorems of Gleason (1957), and Kochen and Specker (1967)? This has been philosophically resolved by arguing for context dependence. But there is a very simple mathematical reason why there is no problem. The momentum used in Bohm's approach using the x-representation is not the eigenvalue of the momentum operator. It turns out to be only the real part of the momentum operator, as we show in (16.55). Thus what Bohm is saying is that the variables (beables) associated with the particle are not the simultaneous eigenvalues of the position and momentum operators. Indeed, the x-coordinate is the eigenvalue of the position operator, but the Bohm momentum is not the eigenvalue of the momentum operator. The eigenvalues are what you find when you measure a quantity. Measurement is participatory and that is what makes the context dependence inevitable. Let me try to bring out the mathematical aspect of these

things and show how they are related to the non-commutative structure of the quantum formalism.

16.4.2 Non-Commutative Algebra and Phase Space

I will try to bring out the ideas in a simple way. In classical mechanics, position and momentum are always well defined so that the concept of a trajectory in phase space presents no difficulty. Thus the picture shown in Fig. 16.3 is unremarkable.

Fig. 16.3. Classical trajectory in phase space

When we come to quantum mechanics, we have serious problems if we assume the properties of particles are only described by the eigenvalues of operators. In the case of the x-representation, we can say the particle has a well-defined position, but its momentum is totally unknown. On the other hand in the p-representation, the momentum is exactly defined but the position is totally unknown. The appropriate representation is defined by the context and thus the context determines which operator is diagonal. In a position measurement, we must diagonalise the position operator. Thus, for a simple one-dimensional discrete space, we obtain

$$S_1 X S_1^{-1} = X_{\text{dia}} = \begin{pmatrix} x_1 & & \\ & x_2 & \\ & & \ddots \end{pmatrix} . \tag{16.37}$$

We then find the particle in one of the boxes in Fig. 16.4. We can say nothing about its momentum.

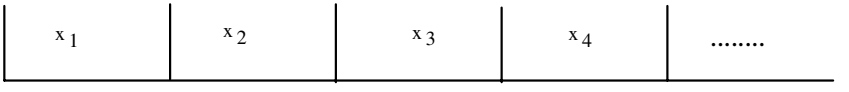

Fig. 16.4. Position measurement in quantum theory

On the other hand, if we perform a momentum measurement, we must diagonalise the momentum operator

$$S_2 P S_2^{-1} = P_{\text{dia}} = \begin{pmatrix} p_1 & & \\ & p_2 & \\ & & \ddots \end{pmatrix} . \tag{16.38}$$

This means that we can only represent the particle in one of the p-boxes as shown in Fig. 16.5, but can say nothing about where it is.

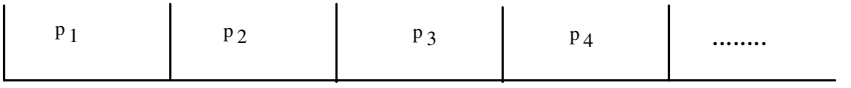

Fig. 16.5. Momentum measurement in quantum theory

Because of non-commutation, $S_1 \neq S_2$, so we can never represent the particle on a phase space defined in terms of the eigenvalues of the X- and P-operators. This is essentially the implication of the work of Kochen and Specker (1967).

The problem arises from the fact that we have given primary physical relevance to the eigenvalues and not the operator algebra. Rather than using eigenvalues, we should let the algebra determine the phase space. We should not start with something we cannot observe (i.e., the phase space) and then build a structure up on that. The key question is thus: Is it possible to start with the algebra of functions and then deduce the properties of the underlying space, if indeed there is one?

16.4.3 Construction of Phase Spaces

The traditional way of building a theory is to start with a topological or a metric underlying space and then to construct on it functions that can form an algebra on that space. However, Gel'fand has shown us that it is possible to start from a commutative algebra of functions and then to abstract any underlying space (for details see Demaret et al. 1997). For example, if we have a commutative algebra of regular functions, $C^\infty(M)$, then the underlying structure is the affine space over the complex numbers. In this space the points are the maximal ideals of the algebra $C^\infty(M)$. If the algebra is a commutative C^* algebra, then we can abstract out a compact topological space, and so on.

The algebraic structure of observables in quantum mechanics therefore seems to offer the possibility of supplying a way to construct an underlying phase space. The problem is that the algebra of observables is non-commutative. With a non-commutative algebra things are not as straightforward as in the commutative case. It is not possible to find a unique underlying space. The best we can do is to construct 'shadow' spaces. Indeed, this is just what the above discrete structure indicates. But here we have only presented

'half' the space. We have no way of representing the complementary property. Thus we can either construct a position space in which there is no momentum specified or specify a momentum space in which case there is no position defined. But we can do more than this. We can actually construct shadow phase spaces, of which the Bohm approach provides one such space. In the rest of this paper, I will show exactly how we can construct such a phase space starting from the algebra.

16.5 The Algebraic Approach

16.5.1 The Extended Heisenberg Algebra

We have already seen in the last section that quantum mechanics 'lives' in the covering space of symplectomorphisms. We now argue that a more general way to look at this covering space is in terms of the Heisenberg algebra or at least, a generalisation of it. The equation of motion in this space is normally taken to be the Heisenberg equation

$$\mathrm{i}\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} + \left[\hat{H}, \hat{A}\right] = 0 \,. \tag{16.39}$$

However, we will find that this is not the way to proceed. We are not simply interested in looking at the time development of the operators because we are still left with the wave function, $\psi(t_0)$, 'frozen' in time and not part of the algebra. We want to retain the time development in the wave function but we need somehow to 'put' the wave function into the algebra. To do this we must go to a more general description of the notion of a state. The way to do this is to use the density operator even in the case of a pure state. For the purposes of this paper it is not necessary to consider mixed states, but the density operator will nevertheless be our primary concern.

Now we know that the density operator can be represented in a Hilbert space by

$$\hat{\rho} = |\psi\rangle\langle\psi| \,. \tag{16.40}$$

However, we do not want to use the properties of Hilbert space at present. This allows us the generalisation we need. In fact we will follow Emch (1972) and write the density operator as

$$\hat{\rho} = \hat{\psi}_{\mathrm{L}}\hat{\psi}_{\mathrm{R}} \,, \tag{16.41}$$

where $\hat{\psi}_{\mathrm{L}}$ and $\hat{\psi}_{\mathrm{R}}$ are elements of the left and right ideals in the algebra. In the case of the orthogonal Clifford algebra, one can construct these ideals with comparative ease, as has been shown in Hiley (2003). What we need to do is to find a primitive idempotent in the algebra and then build the left

(right) ideals by simply multiplying from the left (right) using elements of the algebra.

When we come to the symplectic Clifford (Heisenberg) algebra, we have a problem. The Heisenberg algebra is nilpotent, which means that the algebra does not contain any idempotents. This does not present us with an insurmountable problem since one can extend the algebra as shown in Frescura and Hiley (1984) and Hiley (2003). But Dirac (1947) already anticipated the need for this generalisation. In effect he added an idempotent to the Heisenberg algebra when he introduced the notion of a standard ket and standard bra. These two objects are relatively unfamiliar objects and are not the usual bra and ket used in everyday physics. They are nevertheless related.

Dirac's idea was essentially to lift the ordinary ket into the algebra, which he did by noting that there is no need to write the original 'bar' in front of the angle bracket. Thus $|\ \rangle$ is replaced by $\rangle$. It is the latter symbol that Dirac called the standard ket. This enabled him to construct left ideals in the algebra symbolically, simply because there is no meaning to multiplication on the right. Conversely, he replaced $\langle\ |$ with $\langle$ to enable the right ideals to be generated. If we put the standard ket together with the standard bra, $\langle$, we have in effect introduced an idempotent. [For an alternative use of these idempotents see Kauffman (2002 and 2002a).] We can thus write $\rangle\langle = \varepsilon$. Then $\varepsilon^2 = \varepsilon$, establishing that it is an idempotent. Hence we can write

$$\rho = |\psi\rangle\langle\psi| \Longrightarrow \hat{\psi}\rangle\langle\hat{\psi} \Longrightarrow \hat{A}\varepsilon\hat{B} = \hat{\psi}_{\mathrm{L}}\hat{\psi}_{\mathrm{R}}\ , \tag{16.42}$$

where $\hat{\psi}_{\mathrm{L}} = \hat{A}\varepsilon$ is an element of the left ideal. This is the operator equivalent to the wave function. In the same way, $\hat{\psi}_{\mathrm{R}} = \varepsilon\hat{B}$ is an element of the right ideal, which is the operator equivalent to the conjugate wave function. The left ideal $\hat{\psi}_{\mathrm{L}}$ is also called the algebraic symplectic spinor, while the right ideal $\hat{\psi}_{\mathrm{R}}$ is known as the dual symplectic spinor. These objects are representation free.

In this approach the Heisenberg equation of motion is replaced by two operator Schrödinger equations:

$$\mathrm{i}\frac{\partial\hat{\psi}_{\mathrm{L}}}{\partial t} = \hat{H}\hat{\psi}_{\mathrm{L}}\ , \quad -\mathrm{i}\frac{\partial\hat{\psi}_{\mathrm{R}}}{\partial t} = \hat{\psi}_{\mathrm{R}}\hat{H}\ . \tag{16.43}$$

Let me stress again that these equations are representation free. Now if we take the difference between these two equations, we get

$$\mathrm{i}\frac{\partial\hat{\rho}}{\partial t} + \left[\hat{\rho}, \hat{H}\right]_{-} = 0\ . \tag{16.44}$$

This will be recognised as the Liouville equation, but written in terms of operators. It can be regarded as an equation governing the time evolution of the amplitude of the process.

The sum of the two equations in (16.43) gives

$$\mathrm{i}\left[\left(\frac{\partial\hat{\psi}_{\mathrm{L}}}{\partial t}\right)\hat{\psi}_{\mathrm{R}}-\hat{\psi}_{\mathrm{L}}\left(\frac{\partial\hat{\psi}_{\mathrm{R}}}{\partial t}\right)\right]=\left[\hat{\rho},\hat{H}\right]_{+}. \tag{16.45}$$

Thus we see that we need both the commutator and the anticommutator to give a complete description of the content of the Schrödinger equation and its dual. An interesting argument as to why both equations are needed is given in Hiley (2001b).

We can considerably simplify this equation if we can polar decompose $\hat{\psi}_{\mathrm{L}}$ and $\hat{\psi}_{\mathrm{R}}$ so that we can write

$$\hat{\psi}_{\mathrm{L}}=\hat{R}\hat{U}\,,\qquad\hat{\psi}_{\mathrm{R}}=\hat{U}^{\dagger}\hat{R}\,,$$

where $\hat{R}$ is positive definite and $\hat{U}$ is unitary. Then we find

$$\mathrm{i}\left[\left(\frac{\partial\hat{\psi}_{\mathrm{L}}}{\partial t}\right)\hat{\psi}_{\mathrm{R}}-\hat{\psi}_{\mathrm{L}}\left(\frac{\partial\hat{\psi}_{\mathrm{R}}}{\partial t}\right)\right]=\mathrm{i}\hat{R}\left[\frac{\partial\hat{U}}{\partial t}\hat{U}^{\dagger}-\hat{U}\left(\frac{\partial\hat{U}^{\dagger}}{\partial t}\right)\right]\hat{R}=\left[\hat{\rho},\hat{H}\right]_{+}. \tag{16.46}$$

Now suppose we write $\hat{U}=\mathrm{e}^{\mathrm{i}\hat{S}}$, where $\hat{S}=\hat{S}^{\dagger}$. This implies

$$\hat{R}\left(\frac{\partial\hat{S}}{\partial t}\right)\hat{R}+\frac{1}{2}\left[\hat{\rho},\hat{H}\right]_{+}=0\,. \tag{16.47}$$

If we further assume that $\left[\hat{R},\partial\hat{S}/\partial t\right]=0$, then this equation simplifies to

$$\hat{\rho}\left(\frac{\partial\hat{S}}{\partial t}\right)+\frac{1}{2}\left[\hat{\rho},\hat{H}\right]_{+}=0\,. \tag{16.48}$$

We see that this is simply an equation that describes the time development of the phase operator. Thus we see that the two Schrödinger-type equations (16.43) are replaced by the two equations (16.44) and (16.48), viz.,

$$\mathrm{i}\frac{\partial\hat{\rho}}{\partial t}+\left[\hat{\rho},\hat{H}\right]_{-}=0\,,\quad\hat{\rho}\left(\frac{\partial\hat{S}}{\partial t}\right)+\frac{1}{2}\left[\hat{\rho},\hat{H}\right]_{+}=0\,.$$

Notice that (16.44) uses the commutator, while (16.48) is expressed in terms of the anticommutator (or Jordon product). These equations were originally derived in Brown and Hiley (2000).

16.5.2 Relation to the Bohm Approach

I now want to relate this algebraic approach to the Bohm approach. To bring out this connection, first note the similarity between the pair of defining equations (16.44) and (16.48) and the pair of equations that form the basis of

the Bohm approach, (16.31) and (16.32). The first difference is that equations (16.44) and (16.48) are operator equations and, as we have remarked earlier, they are representation free. On the other hand, (16.31) and (16.32) are tied to a specific representation, namely, the position representation. In fact, as we show below, (16.44) and (16.48) produce the two Bohm equations if we go to the x-representation. This may be surprising since (16.32) contains the quantum potential, whereas (16.48) has nothing that remotely looks like a quantum potential.

In order to show why these two equations contain the same information, we need to project (16.44) and (16.48) into a specific representation. We will first do this in a general representation defined by

$$\hat{A}|a\rangle = a|a\rangle \ . \tag{16.49}$$

Then immediately we find that (16.44) becomes

$$\mathrm{i}\frac{\partial P(a)}{\partial t} - \left\langle \left[\hat{H}, \hat{\rho}\right]_- \right\rangle_a = 0 \ , \tag{16.50}$$

where $P(a)$ is the probability of finding the particle in $|a\rangle$. Thus (16.50) is just the Liouville equation, which expresses the conservation of probability. This can be easily shown if we choose the Hamiltonian to be $H = p^2/2m + V$ and replace a by x, i.e., we choose the x-representation. In this representation, (16.44) becomes identical to the Bohm equation (16.31).

In a general representation, (16.48) becomes

$$P(a)\frac{\partial S(a)}{\partial t} + \frac{1}{2}\left\langle \left[\hat{H}, \hat{\rho}\right]_+ \right\rangle_a = 0 \ , \tag{16.51}$$

which is clearly an equation for the time development of the phase equation, thereby justifying our original claim.

Equation (16.51) looks remarkably similar to (16.32), but again it seems as if the quantum potential is missing. However, the quantum potential is actually implicit in the anticommutator. To bring this out, we have to choose a particular Hamiltonian. For reasons that will become clear, this time we will choose the Hamiltonian for the harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{Kx^2}{2} \ .$$

Substituting this in (16.51) using the x-representation gives

$$\frac{\partial S_x}{\partial t} + \frac{1}{2m}\left(\frac{\partial S_x}{\partial x}\right)^2 + \frac{Kx^2}{2} - \frac{1}{2mR_x}\left(\frac{\partial^2 R^2}{\partial x^2}\right) = 0 \ . \tag{16.52}$$

We immediately see that the quantum potential has appeared, giving us an expression for the conservation of energy. If we do the same thing in the p-representation, we find

$$\frac{\partial S_p}{\partial t} + \frac{p^2}{2m} + \frac{K}{2}\left(\frac{\partial S_p}{\partial p}\right)^2 - \frac{K}{2R_p}\left(\frac{\partial^2 R_p}{\partial p^2}\right) = 0\,. \tag{16.53}$$

A quantum potential appears once again. We can thus construct a Bohm interpretation in the p-representation as well. Here (16.53) is again an expression for the conservation of energy. We can check this simply by considering the ground state of the harmonic oscillator and showing that both (16.52) and (16.53) give the well-known result $E = \omega/2$.

All this shows that we can construct a Bohm interpretation for any representation whatsoever. In fact, mathematically, we have not lost the symmetry between x and p that Heisenberg (1958) and others complain about. The full symplectic symmetry is still there and shows that the singling out of the x-representation is not made on mathematical grounds. It is made on other grounds. Indeed, as Bohm remarked to me once, it seemed more natural to choose the x-representation because all our experiences are in spacetime. I would add to this and argue that our measurement instruments separate different results by positions of pointers, etc. Even digital displays of instrument readings are ultimately displayed in spacetime. Thus in practice we use spacetime to make the physical properties of any system manifest. Further details of this approach appear in Hiley (2002).

16.5.3 Detailed Comparisons

In order to interpret (16.48) as a conservation equation, not only do we have to identify Q_x with a new quality of energy, we also have to interpret $\partial S_x/\partial x$ as the momentum, p. This is, of course, just what is sometimes known as the 'guidance condition', which Bohm (1953) himself took as a subsidiary condition. Notice that, to construct our phase space, we must abandon the insistence that only the eigenvalues of operators have physical meaning.

Now in the light of the results of Sect. 16.3, we can see how the momentum relation arises naturally from the classical limit

$$S_x \longrightarrow S_{\rm cl}\,, \quad p \longrightarrow p_{\rm cl} = \frac{\partial S_{\rm cl}}{\partial x}\,, \quad Q \longrightarrow 0\,. \tag{16.54}$$

Since we now have the possibility of a p-representation, we can ask what replaces the guidance condition? Clearly, (16.53) shows that we must replace $-\partial S_p/\partial t$ with x. The resulting equation $x = -\partial S_p/\partial t$ clearly cannot be interpreted as a guidance condition and must be regarded as a subsidiary condition, as originally suggested by Bohm himself (Bohm 1953). We can see that these conditions mean that we are only using the real parts of the momentum and position, respectively. In fact, in the x-representation, we have

$$\frac{{\rm Re}\,[\psi^*(x)P\psi(x)]}{|\psi(x)|^2} = \frac{\partial S_x}{\partial x} = p\,, \tag{16.55}$$

while in the p-representation, we have

$$\frac{\mathrm{Re}\,[\psi^*(p)X\psi(p)]}{|\psi(p)|^2} = -\frac{\partial S_p}{\partial p} = x\ . \tag{16.56}$$

The fact that we use the real part in both these expressions should not be too surprising since, to obtain the Bohm equation (16.32), we had to take the real part of the Schrödinger equation under polar decomposition. Furthermore, it is now clear why the quantum potential is not ad hoc but a necessary feature. It is necessary to ensure that both energy and momentum are conserved. For example, the kinetic energy used in (16.52) is calculated from the real part of

$$\frac{1}{2m}\left[\psi^*(x,t)\hat{P}\psi(x,t)\right]^2\ ,$$

which is clearly not the quantum kinetic energy calculated from

$$\frac{1}{2m}\psi^*(x,t)\hat{P}^2\psi(x,t)\ .$$

In fact the difference is simply the quantum potential. Thus the conservation of energy is just

$$\frac{\partial S_x}{\partial t} + \frac{p^2}{2m} + \frac{K}{2}x^2 - \frac{1}{2mR_x}\left(\frac{\partial^2 R_x}{\partial x^2}\right) = 0\ .$$

In the p-representation, the potential energy which is calculated from the real part of $\left[\phi^*(p,t)\hat{X}\phi(x,t)\right]^2$ cannot be the total potential energy, which must be calculated from $\phi^*(p,t)\hat{X}^2\phi(x,t)$. Once again, the difference is taken care of in the quantum potential Q_p. Thus the conservation of energy demands that we write

$$\frac{\partial S_p}{\partial t} + \frac{p^2}{2m} + \frac{K}{2}x^2 - \frac{1}{2R_p}\left(\frac{\partial^2 R_p}{\partial p^2}\right) = 0\ .$$

Thus I stress yet again, the quantum potential is a necessity and not ad hoc. The criticisms made by Heisenberg (1958) and Polkinghorne (2002) are based on a misunderstanding of the quantum formalism itself.

16.5.4 Shadow Phase Spaces

Now we can see how the Bohm approach can be understood in terms of non-commutative geometry. If we give primary significance to the algebra of the operators, then we cannot construct a phase space with the eigenvalues of the X- and P-operators. This is the position adopted in standard quantum mechanics. However, we do not have to stop there. We can adopt the Gel'fand approach and project from the algebra onto shadow phase spaces. Thus in

the two representations we have discussed, we can construct two different but related phase spaces, both of which have trajectories as shown in Fig. 16.6. Of course, these trajectories are just the streamlines of the corresponding probabilities, as has been shown in detail in Brown and Hiley (2002). In that paper it is shown that the trajectories are nothing more nor less than the probability currents. Thus the basic physical premise of the Bohm approach is simply to assume that, if we retain the notion of a localised particle, then we can regard the particle as following the streamlines of the probability current, an assumption that is actually often made in physical problems in condensed matter physics, particularly in superconductivity problems (see, for example, Feynman et al. 1965).

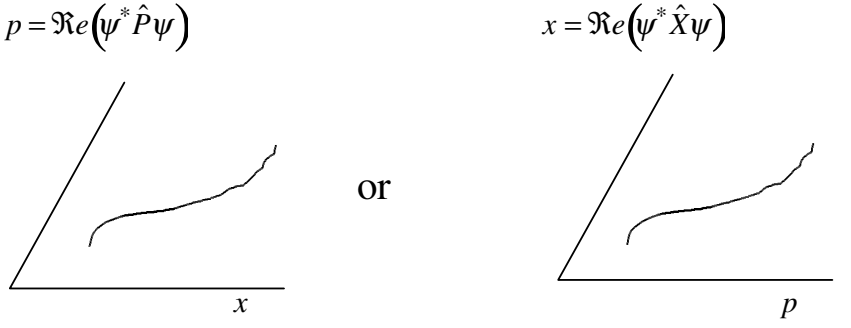

Fig. 16.6. Trajectories in two different phase spaces

16.6 Conclusion

In this paper I have tried to bring out the central role of the symplectic symmetry in both classical and quantum mechanics. Since the algebra of dynamical functions in classical mechanics is commutative, there is no need to distinguish between dynamical operators and their eigenvalues. In quantum mechanics, on the other hand, the algebra of dynamical operators, which carry the symplectic symmetry, is non-commutative. This means, in particular, that we cannot build an x–p phase space out of the eigenvalues of these operators. Indeed, these eigenvalues do not directly satisfy the symplectic symmetry. The symmetry of the eigenvalues is enfolded or implicit in the symplectic symmetry of the operators.

Similarity transformations of the type used in (16.37) and (16.38) show that each single position eigenvalue is 'exploded' into every momentum eigenvalue (and vice versa) under the transformation. This is the source of the probability in the quantum formalism. As long as we insist on identifying the physical properties with these eigenvalues, we will always have this type

of process occurring. It was this specific mathematical structure that Bohm (1980) had in mind when he proposed the notion of the implicate order. In this view the non-commutative algebra is the implicate order. Each measurement makes manifest or explicates an eigenvalue that was enfolded in the algebra of operators. This is the origin of the notion of the implicate order. Each representation therefore produces a specific explicate order. This is just another way of expressing what Pauli (1979) put in a more dramatic way:

> One can look at the world with the p-eye or with the x-eye, but if one wishes to open both eyes at the same time, one goes wrong.

It is not that 'one goes wrong', it is just that the nature of quantum processes is such that it is not possible to manifest both aspects at the same time.

If we want to try to construct a phase space while the algebra of dynamical operators is taken to be primary, then we must resort to constructing shadow manifolds. But this entails giving up the demand that the physical properties must always be characterised by their eigenvalues. We keep the idea that properties made manifest in a measurement are the eigenvalues of the appropriate operator, say, $\hat{A}$. But the complementary variable is not an eigenvalue of the complementary operator $\hat{B}$. Here we assume $\left[\hat{A}, \hat{B}\right] \neq 0$. Instead of eigenvalues, we use the real part of quantities of the type defined in expressions of the form $\mathrm{Re}\langle a|\hat{B}|a\rangle$. If these expressions are used in Schrödinger's equation, then in order to ensure conservation of energy, they must be accompanied by a supplementary energy. It is this energy that has traditionally been called a quantum potential. We see that the quantum potential energy is an internal energy. Thus we can now see exactly why this potential energy is totally different from a classical potential and why it has no external source (for detailed properties of the quantum potential, see Bohm and Hiley 1993).

In this way we see that the Bohm approach is deeply embedded in the quantum formalism and should not be emotionally dismissed out of hand as I have often found it to be. It should provoke no ideological battle. After all, even though we cannot show empirically that a particle actually follows a trajectory, it is equally true that we cannot empirically demonstrate that it does not follow a trajectory. No experiment can decide between these two possibilities, so that the question cannot be resolved by experiment. Surely the Bohm approach simply provides another perspective on quantum processes in general and should be used if it helps to clarify things. It does not usurp the standard approach, it merely shows that there is another way of looking at quantum phenomena which many find useful.

Acknowledgments

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17 Quantum Phenomena Within a New Theory of Time

Avshalom C. Elitzur and Shahar Dolev

A few difficulties with the way present-day physics accounts for time are pointed out, and several novel quantum-mechanical results are described. On the basis of these results, an outline for a new interpretation of QM is proposed, starting from the assumption that spacetime itself is subject to incessant evolution.

17.1 Introduction

In a few crucial passages in the history of physics, seemingly unrelated riddles turned out to merely reflect different facets of the same phenomenon. Such, we submit, may be the lot of the quantum oddities on the one hand, and the elusive nature of time on the other.

Our path to this hypothesis went through pondering several physical issues, which we shall recount in the following sections before describing our findings and proposing our theory. Section 17.2 briefly introduces the two old enigmas of time's apparent transience and asymmetry. Sections 17.3–17.5 point out a simple argument for an intrinsic time arrow. Section 17.6 briefly introduces the advanced action interpretations of QM and their implications. Sections 17.7–17.8 describe some novel experiments that seem to indicate that the wave function evolves in a way that defies ordinary notions of space and time. Section 17.9 proposes an interpretation of these findings, which we broaden to a sketchy outline of a new theory of spacetime in Sect. 17.10.

17.2 Two Peculiarities of Time: Transience and Directionality

Ordinary experience notoriously clashes with physical theory with respect to time. We keep feeling that time 'goes by', that there is a special 'now' moving from past to future, and that future events are born anew out of the present. These characteristics of reality are known as 'becoming'. Yet theoretical physics dismisses this natural impression as mere illusion, and for good reasons. Time is the *parameter* of all motion and change; ascribing motion

or change to time itself is bound to run into absurdities. For example, if time flows, or if the 'now' moves, how fast is this motion? To apply such terms to time would entail a higher time parameter, which would in turn necessitate a yet higher time and so on *ad infinitum*. The vast literature on this issue (see, e.g., [1, 2] and references therein) makes it clear why the overwhelming majority of physicists have avoided this line of thinking altogether, opting instead for the simple and self- consistent 'tenseless' account, which has culminated in relativity theory. Time, by this account, constitutes the fourth dimension, alongside the three spatial ones, of spacetime. All events – past, present and future – coexist along time, just as different sites coexist along space.

It should be stressed that this 'block universe' picture is not just an interpretation of relativity theory but an integral part of it,[1] for even familiar relativistic effects such as length contraction entail it. Consider the following exercise, which may be regarded as the spatial analogue of the 'twin paradox'. A spaceship of length L_0 passes at near-c velocity through a space tunnel of the same length. From the tunnel's reference frame, the spaceship's length is contracted to

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}} \,, \tag{17.1}$$

due to its motion. Hence there is a certain time interval during which the tunnel's two gates can be briefly shut while the entire spaceship travels within the tunnel. From the spaceship's reference frame, however, it is the tunnel that contracts, hence at no time can the entire spaceship reside within it, let alone with the two gates shut!

The two conflicting accounts are compatible only because the two events:

- the entering of the rear of the spacecraft through the tunnel entrance, followed by the entrance gate shutting,
- the emergence of the front of the spacecraft from the tunnel exit, following the opening of the exit gate,

occur in opposite time sequences for the two reference frames (Figs. 17.1 and 17.2).

In the relativistic framework, then, the only objective elements are the two *worldlines* of the spacecraft and the tunnel, extending from past to future, while the 'now' plane is observer-dependent.[2] This 'tenseless' picture is even more pronounced in general relativity, where the reciprocal effects of mass and spacetime on one another presuppose the objective existence of a

[1] Even Einstein himself [3, p. 151] regarded the absence of the moving 'now' in his theory as "a matter of painful and inevitable resignation."

[2] Notice that even the temporal 'twin paradox', when resolved within special relativity without appeal to acceleration, is achieved by employing different 'now' planes for the two observers [4].

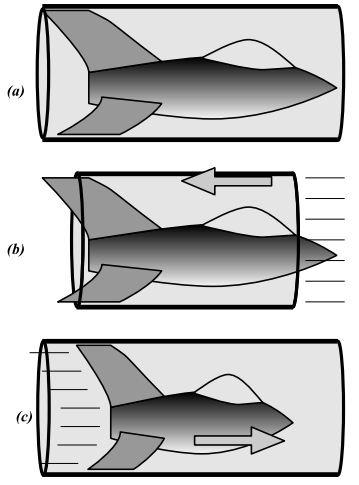

Fig. 17.1. Different relativistic reference frames give different accounts. (**a**) The spaceship and the tunnel are of equal length when in relative rest. (**b**, **c**) Conflicting accounts arise due to the relative motion of the spaceship and tunnel

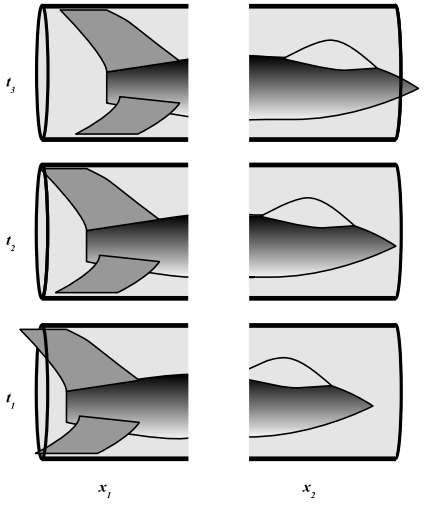

Fig. 17.2. From what seems to be an objective four-dimensional set of events, the reference frame of the spaceship picks (x_1, t_1) and (x_2, t_3) as simultaneous, while for the tunnel (x_1, t_3), (x_2, t_1) are simultaneous

4D spacetime. Time's geometric aspect is even more strongly pronounced in several exotic solutions of relativity that allow spacetime tunnels and closed timelike loops. Relativity, then, allies with basic logic in dismissing the passage of time.

Mainstream physics is similarly dismissive towards the other distinctive attribute of time, namely, its apparent *directionality*. In this respect too, time differs radically from the spatial dimensions: There is no universal 'south' or 'up'.[3] Not so with time, as 'past' and 'future' differ significantly everywhere in the universe due to the second law of thermodynamics. Most physicists, however, belittle this directionality by pointing out that physical law itself is T-invariant. Hence, as nearly all microscopic interactions are time-symmetric,[4] the second law is often denied the status of a real law. Irreversibility, so goes the argument, occurs only in *ensembles* of particles, so it may merely reflect the universe's initial state, which, for some reason, happened to be highly ordered. One could equally conceive of a universe whose initial state was totally disordered but which, by the same T-invariant laws, gradually becomes *ordered* in time.[5]

It is a very impressive feature of the position of mainstream physics that these two negative assertions:

- dismissing time's passage as illusion,
- dismissing time's arrow as an artifact of the initial conditions,

neatly accord with one another. If the universe is a four-dimensional collection of equally-existent events, with no privileged 'now', then both readings of its history are equally valid. Entropy increases *as well as decreases* with time, depending on whether the observer chooses to read the history of the universe forwards or backwards! Whether one likes this account or not, it is admittedly coherent and paradox-free.

Yet, a few dissenting voices are heard, most notably Davies [6], and for convincing reasons too. To believe that even our *future* events, including everything we may decide to do, 'already' exist in time, just as other places exist in space, is very awkward. True, intuition has often been proved deficient by modern physics, but it should not be dismissed off-hand. Our immediate perception of time might be directly sensing an inherent feature of it that has not yet found its place in the formalism. Even relativity theory indicates that time differs from the spatial dimensions in some as-yet unclear way: it

[3] True, weak interactions do not conserve parity. Therefore mirror images of the same physical process are not always equally probable. Nevertheless, there are still no absolute directions of space.

[4] Only 'nearly all'. The CP violation exhibited by weak interactions entails, by CPT invariance, a basic T violation too. "It is hard to believe," says Penrose [5, p. 583], "that Nature is not, so to speak, 'trying to tell us something' through the results of this delicate and beautiful experiment." Hear, hear!

[5] In fact, due to the ergodicity of physical laws, a universe with any initial condition, given a long enough period of time, will reach both entropy-increasing and entropy-decreasing phases. But the huge amount of astronomical and other scientific observations are all compatible with the proposition that the universe was created some thirteen billion years ago, a period much too short for the above argument.

bears the imaginary sign. Why is t assigned the minus sign, viz.,

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - \Delta t^2 \tag{17.2}$$

in Minkowski's equation? Relativity simply *presupposes* rather than explains this difference between t and its three counterparts.[6]

But if the conventional dismissal of time's directionality is congruent with the dismissal of its transience, would a loophole in the former not challenge the latter? Such a possible loophole is discussed next.

17.3 Indeterminism Entails an Intrinsic Time-Asymmetry

It is embarrassing to observe how rarely the vast literature on time's arrow (see, e.g., [8, 9]) refers to the closely related issue of determinism. Are the basic interactions between particles truly random, or is information always preserved at some smaller level? This issue is crucial, as it has a straightforward bearing on the origins of irreversibility. We shall first point out exactly what bearing it has, and in the next section discuss determinism itself.

Recall again the conventional approach: the second law is not a real law but a mere fact – albeit ill-understood – about the beginning of the universe. According to Price [8, p. 262]: "What needs to be explained is the low-entropy past, not the high-entropy future – why entropy goes down towards the past, not why it goes up towards the future." One could, so goes the argument, conceive of a closed system, such as the entire universe, where the initial conditions lead to increasing *order*. All it takes for such an evolution is that the particles in the system should be pre-arranged with the appropriate precise correlations that would ensure their later convergence into increasingly ordered states.

There is, however, a crucial difference between the normal, entropy-increasing evolution, and the time-reversed, order-increasing evolution. *The latter, not the former, requires an infinitely precise pre-arrangement of all the system's elementary particles*. Consequently, when setting a system to evolve into a lower entropy state, any failure of a state to precisely determine the next state during its evolution will, given sufficiently many interactions between the system's constituents, ruin the increase in order. Boltzmann's entropy measure,

$$S = k \ln W \ , \tag{17.3}$$

[6] The question is better put this way: Why can worldlines extend only in a timelike, never a spacelike fashion? The answer would be that the speed of light must never be exceeded, but as Sudarshan [7] has shown, relativity does not forbid the existence of tachyons, whose worldlines would be spacelike. Why, then, are tachyons never observed?

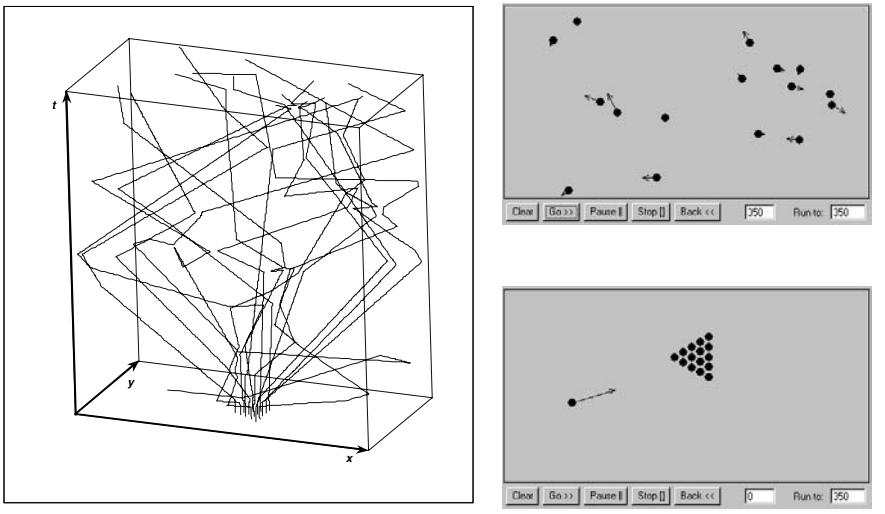

Fig. 17.3. A computer simulation of an entropy-increasing process, with the initial and final states (*right*) and the entire process using a spacetime diagram (*left*). One billiard ball hits a group of ordered balls at rest, dispersing them all over the table. After repeated collisions between the balls, the energy and momentum of the first ball is nearly equally divided between all the balls

is based on the trivial arithmetical fact that there are countless non-special microscopic arrangements that make disordered states while very few, special arrangements make ordered ones. So if nearly every initial arrangement will eventually give rise to entropy increase, any interference following such an initial state is very unlikely to alter this destiny. Not so for the few initial arrangements that lead to order increase. They can give rise to eventual order only if nothing interferes with their later evolution.

Figure 17.3 shows the results of a computer simulation of an ensemble of billiard balls. In the initial state the balls are ordered and all the momentum is concentrated in one ball that hits them. The resulting evolution of the system takes it to a higher entropy state, where the balls are scattered and the momentum is evenly distributed amongst them. Figure 17.4 shows the development of a very unique state. Although it looks disordered, its subsequent evolution will take it to an ordered state. So far, so good. But we now introduce a small random disturbance into the progression of the ordered state. Allowing the entropy-increasing system to evolve (Fig. 17.5), the disturbance only caused an insignificant shift in its destiny from one high-entropy configuration to another, practically indistinguishable one. Not so with the time-reversed process (Fig. 17.6). The slightest variation in the position or momentum of a single particle creates a disturbance in the system's evolution which – given sufficiently many interactions between the particles

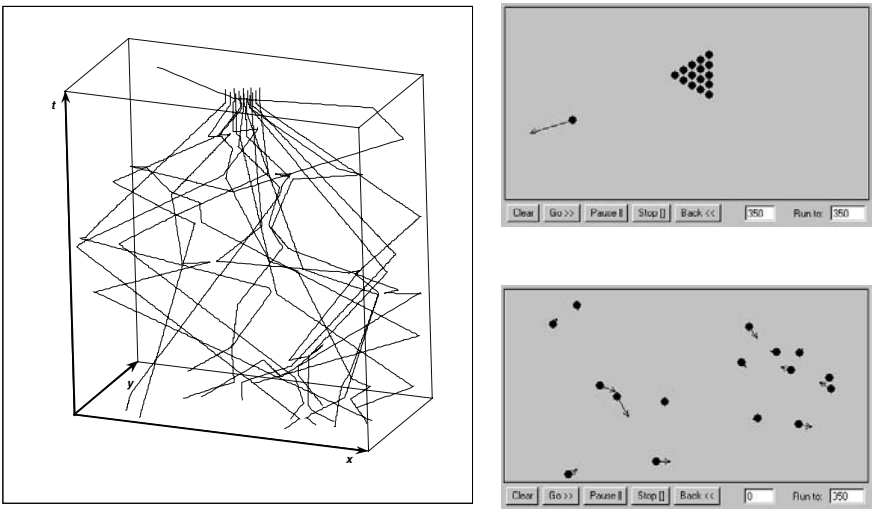

Fig. 17.4. The time-reversed process. All the momenta of the balls are reversed at t_{350}. Eventually, the initially ordered formation is re-formed, as at t_0, ejecting back the ball that initiated the process

– further increases as the system evolves. Consequently, entropy increases in the time-reversed system too.[7]

The relevance of this observation to the origins of irreversibility is immediate [10]: Had physics been able to prove that determinism does not always hold – that some interactions are genuinely probabilistic – it would follow that entropy *always* increases, regardless of the system's initial conditions.[8] An intrinsic time arrow would then emerge in *any* closed system under *whatever* initial conditions, congruent with the time arrow of the entire universe, in which closed systems are supposed to be shielded.

[7] The ergodicity argument can be raised here too. In an indeterministic but ergodic system, after a long enough time, the system will display both entropy increase and decrease. One might therefore argue that the universe's relatively ordered state at the present is due to a mere fluctuation, within which all our scientific observations just happen to comply with a systematic physical theory. We do not need to bother to refute such a possibility as, by the laws of probability, it is susceptible to a powerful *reductio ad absurdum* into solipsism. It is much more probable that it is only the reader's brain state, rather than the entire universe, or even a part of it, that is the result of such a unique fluctuation.

[8] For many years, Hawking has been claiming that unitarity is lost during black hole evaporation, while at the same time maintaining that the thermodynamic asymmetry is only due to the universe's initial condition. We have pointed out the contradiction between these two assertions [11]. Interestingly, Hawking recently recanted his unitarity loss hypothesis [12].

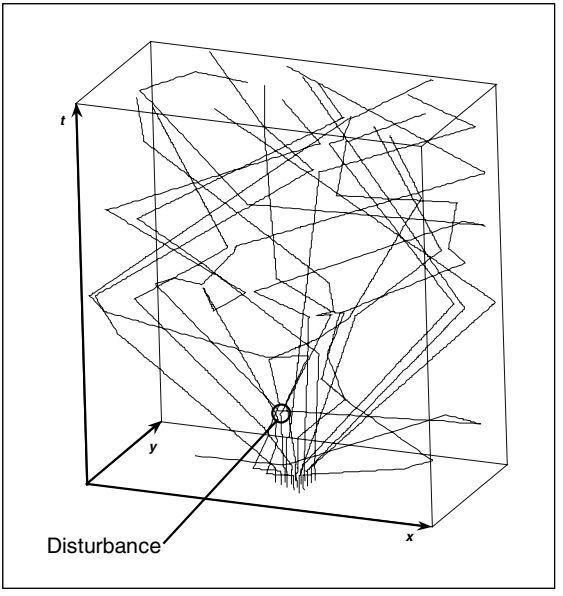

Fig. 17.5. The same simulation as in Fig. 17.3, with a slight disturbance in the trajectory of one ball (marked by the *small circle*). The entropy increase seems to be indistinguishable from that of Fig. 17.3

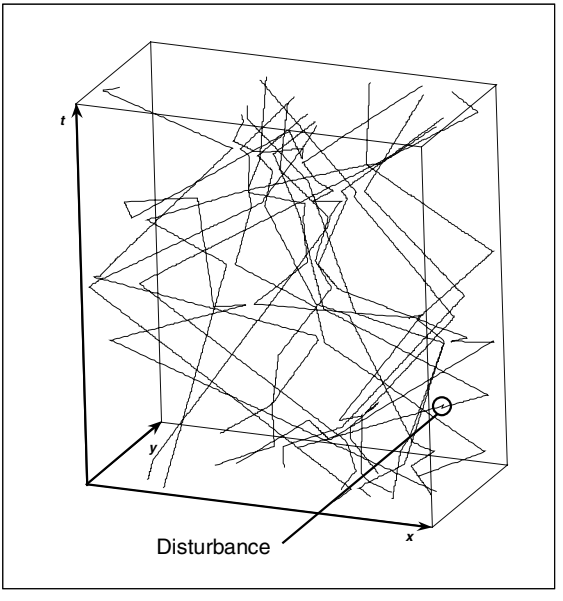

Fig. 17.6. The same computer simulation as in Fig. 17.4, with a similar disturbance. Here, the return to the ordered initial state fails

17.4 Hidden-Variable Theories Are Forever-Hidden-Variable Theories

In other words, if God plays dice, irreversibility is inherent in nearly any process. But *does* he? QM is the natural place to look for an answer.

The way QM bears on the issue can be summarized with three observations:

- The Schrödinger equation is deterministic and works perfectly well for the pre-measured state.
- It fails miserably once the state is observed, measured or interacts in any other way with the environment, so that the superposition gives way to one out of the many previously possible states.
- This new, 'collapsed' state is not known to be causally determined by the pure state that preceded the instance of measurement.

For example, when the spin superposition

$$\Psi = |\uparrow\rangle + |\downarrow\rangle \tag{17.4}$$

gives way, upon measurement, to either $|\uparrow\rangle$ or $|\downarrow\rangle$, nothing in the original state is known to have determined the outcome. Indeterminism, therefore, seems to sneak in during this transition.

The uncertainty principle further stresses this causal void in the pre-measurement state. Intuitively, a tradeoff like

$$\Delta x \Delta p \geq \hbar/4\pi \ , \tag{17.5}$$

which assigns a constant degree of uncertainty to the measurement of certain pairs of variables, suggests that there is a certain *ontological* indeterminacy, rather than mere *epistemological* ignorance, to many physical variables. Indeed, the double-slit experiment, the best visual demonstration of this position–momentum tradeoff, shows that, when the photon/electron wave function passes through the partition, the position is not merely unknown but, genuinely 'smeared' over space, enabling it to gauge both slits at the same time. Momentum, in turn, is similarly 'smeared' when the position is accurately measured.

Ironically, the discoverer of the uncertainty principle does not appear to have fully grasped its profoundness. The conceptual device known as Heisenberg's microscope [13] turned out to be insufficient for explaining the true nature of the uncertainty. It only showed that the influence of the measurement prevents accurate measurement of the particle's position and momentum at the same time. To see that there is more to quantum uncertainty, consider the EPR–Bell experiment [14]. This setup seems to indicate that the variables are not only unknown but *do not exist* before measurement. The symmetry under rotation of the singlet state,

$$\Psi = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\right), \tag{17.6}$$

implies that the two particles lack definite spin values, not only in the z direction but in all other directions as well. And indeed the experimental violations of Bell's inequality show that the spins of the two particles could not have been fixed prior to the measurement. The simplest conclusion, therefore, is that, if (17.6) does not give any preference for either the spin-up or spin-down outcomes, such a preference simply does not exist. Each particle's spin is probably created *de novo* at the instant of measurement, thereby forcing the opposite direction on the other particle.

Determinism, however, proved to be too precious to be given up by all physicists. A survey of the interpretations of QM [15] shows that about half of the interpretations preserve determinism in some form of hidden variables or parallel universes, which supplement the superposition of (17.4) or (17.6) with some additional variables. These variables are believed to determine nonlocally the results of measurements performed on the particles themselves. Even radical new models, such as those due to 't Hooft (Chap. **??**) and Smolin (Chap. 10), go to great lengths to preserve determinacy by assuming hidden variables of one kind or another.

But can these models be scientifically proved? We have a serious concern that research on this issue might go astray for many years, claiming numerous years of futile labor, while seeking something that may, *a priori*, be undetectable. Consider again the above EPR–Bell proof against local realism. While it has led several authors to abandon the idea of hidden variables altogether, many others (including Bell himself) kept envisioning *nonlocal* hidden variables instead. What these models basically assume is that the two particles leave the source not superposed but with some pre-existing values of the hidden variables which carry on a common context for the spins of both particles. Then, upon measurement of one particle, this shared context affects the result of the measurement performed far away on the other particle. Now, to the extent that these models are fully deterministic, they assume that even this change of the spin, brought about by the measurement, obeys causal laws. But here a simple question ought to be raised: Can such hidden variables ever be observed? A simple analysis can easily show that, *if quantum nonlocality is not buffered by indeterminacy, relativity must be empirically violated*.

This conclusion is quite straightforward, yet the basis for it has seldom been explored. Elitzur [16] has pointed out that the three basic no-no's of theoretical physics:

- the quantum-mechanical impossibility of predicting the outcome of a measurement,
- the relativistic prohibition on superluminal velocities,
- the thermodynamic unlikelihood that the entropy of a closed system will decrease,

intriguingly support one another, in such a way that violation of one principle leads to violation of the other two.

The argument, however, was qualitative, failing to give a rigorous proof. Yet indirect support for it came from a study that made an opposite claim. Valentini [17,18] boldly suggested that the relativistic prohibition on superluminal velocities is merely due to entropy increase at the quantum level which has 'scrambled' the hidden variables of quantum particles, making these variables akin to 'noise'. He went on to suggest that, if a technique were developed to distill a handful of particles in a low-entropy state, these particles could be used, for example, to instantaneously transmit information through the singlet state. The relativistic upper limit on c was thus rendered 'fact-like' rather than 'law-like', just like the second law of thermodynamics. This is a far-reaching hypothesis, with the added merit of being testable. For our purpose it should be noted that it reaffirms that, once quantum nonlocality is not buffered by indeterminacy, violations of relativity are bound to occur.

However, we ourselves believe that the laws of relativity – so simple, coherent, and beautiful – reflect something very profound about physical reality rather than being just a consequence of noise. Likewise for quantum uncertainty: it is more likely to be conveying some fundamental aspect of causality than to be merely reflecting a technical limit of measurement. Indeed, the Bekenstein–Hawking [19–21] and the Unruh [22] effects seem to indicate that QM, relativity and thermodynamics are related in some as-yet unfathomed ways. Most likely, therefore, the next revolution in physics will be a theory that will incorporate relativity and QM as important ingredients.

Therefore, although there is no clear resolution at present to the issue of (in)determinism, our conclusion stands: *For any future theory in which relativity theory will be an integral ingredient, hidden variables must remain forever unobservable*. This places these entities in a position that is much more problematic than that of the ether. A physical theory based on entities, the detection of which is *forbidden* by the theory itself, belongs rather to the realm of religion.[9]

[9] In order to better assess the theoretical impasse involved with hidden variables within a relativistic theory, consider the status of quarks in particle physics. Quarks too cannot be directly observed, due to their confinement. Yet particle physics has pointed out several predictions that follow from the existence of quarks and are unaccountable in any theory that does not make this assumption. These predictions have so far been verified. No such falsifiable prediction is proposed by the hidden variable theories.

17.5 An Interim Conclusion: Time Is Intrinsically Asymmetric

To summarize the issue of time asymmetry, the evidence we can point to is admittedly circumstantial, but seems fairly compelling. It is reasonable to say that:

1. QM implies an indeterminacy in any interaction in which a quantum system interacts with the environment.
2. To the extent that this indeterminacy is only apparent and deeper hidden variables underlie it, then, by relativity theory, these variables must never be detected.
3. A theory based on *absolute* unobservables is unscientific. Indeterminism, therefore, is a simpler, hence perhaps ontologically better description of Nature.
4. But if any measurement-like interaction is truly indeterminate, then an intrinsic time asymmetry, independent of initial conditions, must be inherent to any process in which such an interaction of a quantum system with the environment is present.

This argument for time asymmetry reopens the issue of time transience. If determinism does not hold, then mainstream physics can no longer boast the consistency between denying time transience and dismissing time asymmetry, pointed out at the end of Sect. 17.2. In a universe not strictly governed by determinism, one reading of the universe's history – initial order gradually giving way to increasing entropy – is perfectly reasonable, while the time-reversed account – high entropy gradually converging into order – is absurd or even solipsistic. In other words, in the absence of a proof for determinism, we have no reason to believe that the future 'already' exists, causally determining the universe's present and past. The person-in-the-street picture of becoming, in which the future is ontologically inexistent, to be genuinely created anew, regains credibility.

We shall now put forward some new quantum-mechanical evidence in favor of this apparently naive view.

17.6 The Advanced Action Hypothesis

It is again to QM that we turn in search for new insights into the nature of time. Aharonov ([23] and Chap. 15) and later Cramer [24] proposed two very appealing interpretations of QM (the two-vector formalism and the transactional interpretation, respectively) which, for the purpose of the present discussion, can be taken as one model, henceforth dubbed the Advanced Action (AA) hypothesis. Our use of the adjective 'advanced' is two-fold: it complies with the capricious physical convention that refers to retroactive action as 'advanced' and normal action as 'retarded', and it discloses our personal bias

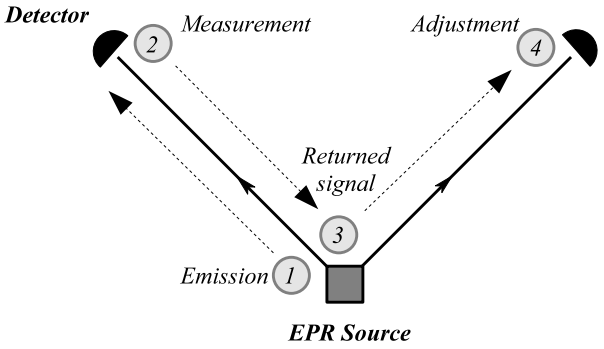

Fig. 17.7. AA in an EPR experiment. After the emission of the particles at the EPR source (1), a measurement occurs at one of the detectors (2). The effect of the measurement then returns back to the EPR source along the past worldline of the particle (3) and from there follows the other particle's worldline to inform it of the change in state (4)

in favour of this idea. The noun 'hypothesis' further conveys the hope that this interpretation may eventually yield testable predictions.

According to AA, any quantum interaction is brought about not by one wave function but by the combined effect of two (or even more) waves, going back and forth in time. The initial wave goes from the source to the future absorber(s), such as measuring device, observer, etc. (one or more), while the reciprocal, 'advanced', waves(s) return to the source backwards in time.

The famous EPR experiment provides a quick demonstration of the elegance of AA. The measurement of one particle affects not only that particle's state at the moment of measurement but also all its previous states – indeed its entire worldline right down to the source – and then zigzags back to the other particle up to the present (Fig. 17.7). Cramer [24] has systematically applied AA to explain a vast range of famous quantum-mechanical peculiarities (see [25] for a recent perspective, and also [26] for some novel information-theoretic advantages of this model).

As revolutionary as AA is, however, Cramer [24] stresses that his interpretation of QM is just that, namely, an interpretation, not a theory, and hence yields just the same predictions as the quantum formalism itself. He furthermore endorses the standard Block Universe picture of time. It is within a 'static' (Cramer's term) four-dimensional spacetime that the mutual 'transactions' between past and future events take place.

In contrast, Aharonov, while not proposing predictions that differ from those of quantum theory, still derives from AA predictions that would probably never have been predicted within another theoretical framework. He is even more unorthodox in his approach to the nature of time, stating – although so far only in personal communication – that his interpretation entails a true dynamics of spacetime itself. He believes that every instant in time

within a quantum process is visited twice: first by the forward propagating wave function and then by the complementary one.

It is here that we would like to go a step further. In the following sections we propose two experiments whose predicted results, as obliged by QM and inspired by AA, strongly clash with ordinary notions of space and time. On the basis of these experiments, we shall endorse Aharonov's vision of AA within a theory that ascribes genuine dynamics to spacetime itself.

17.7 When One Quantum Object Measures Another: Inconsistent Quantum Histories

The oddities of QM, whether in the form of real experiments or conceptual paradoxes, such as the double-slit, delayed-choice, EPR, and Schrödinger's cat, are many and famous. They are paradoxical in that they point out inconsistencies between QM and classical physics, especially relativity. In this section and the next we present a new family of thought experiments[10] that are paradoxical in the deeper sense that they derive from QM an evolution that seems to be inconsistent with itself.

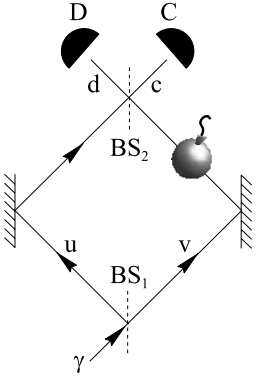

Fig. 17.8. Interaction-free measurement. BS_1 and BS_2 are beam splitters. In the absence of the obstructing bomb, there will be constructive interference on path c (detector clicks) and destructive interference on path d (no click)

One origin of these experiments may be found in Elitzur and Vaidman's [27] Interaction-Free Measurement (IFM) (Fig. 17.8). Using a Mach–Zehnder interferometer (MZI) with an object placed along its v path, EV pointed out

[10] In what follows we shall not bother to distinguish between gedanken and real experiments. QM is so rigorous that no one expects a gedanken experiment not to give the predicted result when performed in reality. And indeed, most of QM's gedanken experiments have by now been successfully performed.

that, in 25% of the cases, a single photon traversing the MZI may end up in detector D, indicating that it has been affected by the object on its v path, and yet that, by the photon's very arrival at BS2, it must have taken the opposite, u path, since otherwise it would have been absorbed by the object. To make things more dramatic, EV took, as the blocking object, a supersensitive bomb that can be detonated by a single photon. Is it possible to know whether the bomb is good without detonating it? Their device allows one to save 50% of the bombs tested this way, a figure later brought close to 100 % by a significant improvement proposed by Kwiat et al. [28], who also carried out the experiment.

The novelty of the EV device lies essentially in an exchange of roles. The quantum object, rather than being the *subject* of measurement, becomes the measuring apparatus itself, whereas the macroscopic detector (or supersensitive bomb in the original version) is the object to be measured. In their paper [27], EV mentioned the possibility of an IFM in which both objects, the measuring and the measured, are also single particles, i.e., quantum mechanical objects, in which case even more intriguing effects can appear.

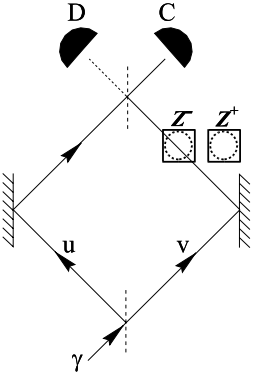

Fig. 17.9. Mutual IFM, where the 'bomb' is also quantum-mechanical

This proposition was taken up in a few seminal papers by Hardy [29–31]. In one of these articles, the bomb has been replaced by another superposed atom. Figure 17.9 illustrates this experiment. A photon traverses an MZI. On one arm of the MZI there is a spin-1/2 atom prepared in a spin state $|X+\rangle$, that is, $\sigma_x = +1$, and split by a non-uniform magnetic field M into its two Z components. The box is then carefully split into two halves, each containing either the $|Z+\rangle$ or the $|Z-\rangle$ part, while preserving their superposition state. In other words, if the atom's spin in the Z direction is 'up' it resides in one box and if it is 'down' it is in the other. The boxes are transparent for the photon but opaque for the atom. The $Z+$ box of the atom is positioned across the photon's v path in such a way that the photon can pass through the box

and interact with the atom inside it with 100% efficiency. Then a photon is sent into the apparatus. Here the photon and the atom, so to speak, measure each other's position.

In 25% of the cases, this mutual measurement will be completed, with the result that the photon took the v path and the atom turned out to be in the intersecting box on that v arm. Hence it will absorb the photon and be in an excited state. Let us discard these cases. In another 50% of cases, a photon will end up in detector C. This group gives no conclusive results, so let us ignore it too.

It is the remaining 25% cases that are the most curious. The photon ends up in detector D, indicating that its v path has been blocked and that it must have taken the u path, but this measurement has also 'collapsed' the atom on the v path. In other words, the atom must always be found in the intersecting box.

Notice that the loss of the atom's superposition is a real physical effect. Prior to the photon passage, the atom's two boxes could be reunited, and the atom's spin state $|X+\rangle$ could be measured and shown to be intact. (This is quite analogous to the interference effect.) Not so after the photon has traversed the MZI! The atom's position in the intersecting box is now certain and its X spin is consequently random. And yet, despite this physical effect exerted on the atom, the photon, which is supposed to have caused this effect, seems to have taken the opposite, u MZI arm!

Stimulated by this result due to Hardy, we began devising other experimental setups in which several particles 'measure' one another before the macroscopic detector completes the measurement. The result is a few experiments in which the history they yield seems to be inconsistent.

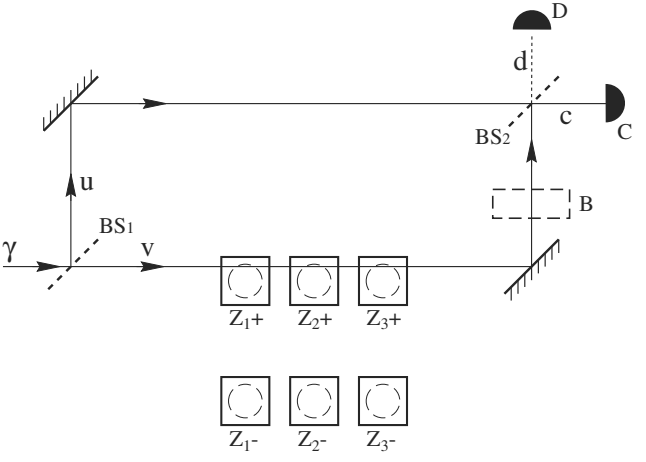

Fig. 17.10. One-photon MZI with several interacting atoms

For example, we have replaced Hardy's atom on path v in Fig. 17.10 by a row of superposed atoms. The result predicted by QM is that only *one* of the atoms, *not necessarily the first*, will lose its superposition, while all the others will remain intact. In other words, all atoms on path v except one will preserve their x spin when reunited [32].

Did one of the atoms in the row block the photon's way on path v? No, because if one places an opaque object at the end of the atom row (object B in Fig. 17.10), and no atom has absorbed the photon, *all* atoms will remain superposed! It seems that something must have passed through the row after all.

How can the photon wave function affect only one out of many atoms positioned in a row along its path, leaving the others apparently intact, and yet complete its way through the row to the BS? Naturally, any answer to this question is bound to be controversial, as the Copenhagen, Pilot Wave, Many Worlds and other interpretations would propose different explanations. One lesson, however, might be accepted by the majority of physicists: *Measurement affects not only the system's present state but its entire history.*[11] It is the final click at d that seals the process. This is, in fact, the lesson derived from Wheeler's delayed choice experiment [33]. Wheeler himself chose to interpret it by strict adherence to the Copenhagen interpretation: "No phenomenon is a phenomenon until it is an observed phenomenon." But perhaps it is time we did not shy away from an ontological conclusion, namely, that a measurement at the end of a quantum process genuinely affects the history of the process in both directions of time.

17.8 The Quantum Liar Paradox

If measurement can sometimes 'rewrite' the history of a quantum process, some traces of this 'rewriting' may be found in the form of odd inconsistencies within the resulting history. In terms of footnote 11, a scenario is possible which is analogous to a Schrödinger cat found to be long dead alongside with the scratches and droppings within the box which indicate that it has been alive all that time.

Consider two atoms in the $|X+\rangle$ state, each separated according to its Z spin into two boxes as in the previous section (Fig. 17.11). Two coherent laser beams are directed towards an equidistant beam-splitter (BS), behind which

[11] One of us (AE) owes this insight to a student's question about Schrödinger's cat. She argued that, if the box is opened after sufficiently many hours, it should be possible to know whether the cat has been dead or alive during the preceding hours. If the cat has been alive all that time, it will have soiled the box and left scratches on its walls, whereas if it has been dead, it will show signs of decomposition. Here too, the measurement at the moment of opening the box must select not only the cat's state at that moment but its entire history within the box.

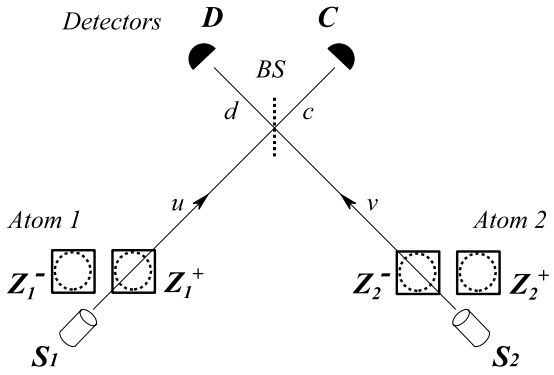

Fig. 17.11. Entangling two distant atoms that have never interacted

are two detectors. Each beam crosses one of the two boxes of either atom. The laser sources are of low enough intensity to ensure that, on average, only one photon is emitted during a given time interval t. When the atoms are not present, the two laser sources are set to interfere constructively on branch c and destructively on branch d. This coherency can last for a period of time $\tau \gg t$. Notice the oddity of the situation: a single photon is detected at C, yet by QM, the very uncertainty about its origin makes it interfere constructively, as if it has originated from the two sources!

Now consider the case in which the two atoms are present and detectors D click. We know that one of the beams was blocked (thereby spoiling the destructive interference). This means that one of the atoms 'collapsed' into the intersecting box, and the other into the non-intersecting one, although we do not know which atom collapsed into which box. Again this uncertainty suffices to entangle the two atoms into an EPR state [34]:

$$|\Psi\rangle = |Z+\rangle_1 |Z-\rangle_2 - |Z-\rangle_1 |Z+\rangle_2 \,. \tag{17.7}$$

Notice that this experiment may be regarded as a time-reversed EPR, as the two atoms do not share a common event in the past but rather in their future, so to speak. It will therefore be referred to as RPE henceforth.

But the most intriguing feature of the experiment emerges once we employ the famous tool for proving a nonlocal influence between entangled particles, namely, Bell's inequality. Let us first recall the gist of Bell's nonlocality proof for the ordinary EPR experiment [14]. Let a pair of EPR particles be created with total spin zero. Let the two particles travel to two equidistant measuring instruments. Now consider three spin directions, x, y, and z. On each particle of the pair, a measurement of one of these directions should be performed at random. Let many pairs be measured this way, such that all possible combinations of x, y, and z measurements are eventually performed. Then let the incidence of correlations and anti-correlations be counted. By quantum mechanics, all same-spin pair measurements will yield 100% correlations, while

all different-spin pair measurements will yield non-correlated results (half correlated and half anti-correlated). And indeed, this is the result obtained by numerous experiments to this day, e.g., [35–38] to name but a few. By Bell's proof, such a unique combination of correlations and anti-correlations cannot have been pre-established. We conclude that *the spin direction (up or down) of each particle is determined by the choice of spin angle (x, y, or z) measured on the other spacelike-separated particle, no matter how distant.*

This is the familiar EPR–Bell scenario. Let us now apply this method to RPE. Recall that each atom has been split according to its spin in the z direction. Therefore, to perform the z measurement, one has simply to open the two boxes and check where the atom is. To perform x and y spin measurements, one has to reunite the two boxes under the inverse magnetic field, and then measure the atom's spin in the desired direction. Having randomly performed all nine possible pairs of measurements on the pairs, many times, and using Bell's theorem, one can prove that the two atoms affect one another instantaneously, as in the ordinary EPR, with the difference that they share an event not in the past but in the future.

However, a puzzling situation now emerges. In 44% (i.e., 4/9) of the cases (assuming random choices of measurement directions), one of the atoms will be subjected to a z measurement – namely, checking in which box it resides. Suppose, then, that the first atom was found in the intersecting box. This seems to imply that *no photon has ever crossed that path, which is obstructed by the atom*. But then, by Bell's proof, the other atom is still affected nonlocally by the measurement of the first atom. But then again, if no photon has interacted with the first atom, the two atoms share no causal connection, in either past or future!

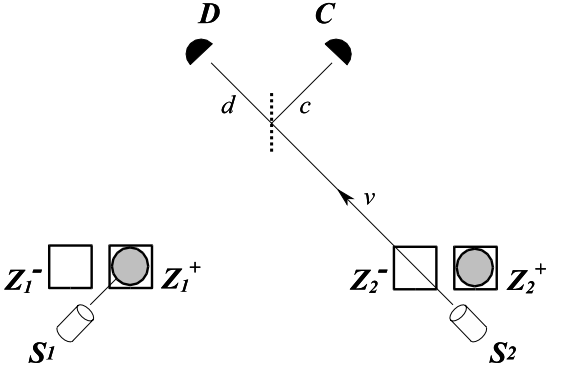

Fig. 17.12. Entangling two atoms

The same puzzle appears in the cases in which the atom is found in the non-intersecting box. In this case, we have a 100% certainty that the

other atom is in the intersecting box, meaning, again, that no photon could have taken the other path. But here again, whether we subject the other atom to the 'which box' measurement or to an x or y measurement, Bell-inequality violations will occur, indicating that the result was affected by the measurement performed on the first atom (Fig. 17.12).

Put otherwise, *the very fact that one atom is positioned in a place that seems to preclude its interaction with the other atom is affected by that other atom*. This is logically equivalent to the statement: "This sentence has never been written." We are unaware of any other quantum mechanical experiment that demonstrates such inconsistency.[12]

17.9 A Hypothesis: The Quantum Interaction Involves a 'Rewriting' of the Evolution in Spacetime

Although the existing interpretations of QM will claim that they have no difficulty in explaining the above results, our search for a model that will be, at the same time, realistic, parsimonious and, if possible, elegant, has led us to propose an interpretation of our own. We aspire to deal with the oddities of QM not by abandoning the hypothesis of objective reality existing out there, but by working within a realistic framework that forces one to propose new hypotheses that may later be subjected to empirical test. We also seek to integrate the four-dimensional spacetime of relativity with the somewhat opposite hints provided by QM that genuine change, not static geometry, is the most basic property of reality.

General relativity has taught us that spacetime is a real physical entity, namely, a four-dimensional manifold of worldlines with their corresponding curvatures. Within this geometric picture, the transactional interpretations reviewed in Sect. 17.6 fit in very naturally, as they require interactions between earlier and later events. Where we break new ground is in proposing that this spacetime is not the changeless Minkowski array. Perhaps rather spacetime itself is subject to evolution. True, ascribing evolution to spacetime itself runs the risk mentioned in Sect. 17.2, i.e., of invoking an infinity of higher- and higher-order times. We shall face this concern in the next section.

[12] It is possible to make this experiment even more striking by entangling two excited atoms, out of which only one can emit a photon within a given time interval. The atoms thus become entangled with respect to their excited/non-excited state. A Bell-type inequality can be formulated for this case by using measurements that are orthogonal to the excited/non-excited state. Here too, the measurement of one atom may show it to be excited, thereby making it appear as if it has never emitted a photon, and thus could never become entangled with the other atom. And yet, by Bell's inequality, this result must also be affected by measurement of the other atom. We are currently elaborating such an experimental scheme.

If this is so, if spacetime itself evolves, then experiments yielding apparently inconsistent histories, like those described above, may warrant an account that goes something like this: first a retarded interaction brings about history $t_1x_1, t_2x_2, \ldots$, and then an advanced interaction transforms this history into $t_1x'_1, t_2x'_2, \ldots$. Consider the above quantum liar paradox. Perhaps there was first a forward moving evolution by which the two atoms sent virtual photons towards the BS. Then the detection of one photon retroactively entangled the two atoms backwards in the past. Finally, the measurement of one atom, which found it to be in the intersecting box, obliterated all traces of its interaction with the rest of the experimental setup. These reiterations of the process occurred by repeated spacetime zigzags à la Aharonov and Cramer, but in some real, higher time dimension, over spacetime itself. Such a model may also be better capable of explaining a few other surprising results discovered lately by similar methods [40, 41].

17.10 An Outline of the Spacetime Dynamics Theory

Our study was motivated by two phenomena which, on the one hand, have no trace in physical law, and, on the other hand, seem to constantly proclaim their presence:

- Time, unlike space, seems to be flowing. However, accepting this phenomenon as a true property of time entails several logical and physical difficulties, such as an endless series of time parameters, so it has often been dismissed in favor of the simpler, self-consistent Block Universe picture.
- The fact that we never observe the superposed states of the microscopic world in our macroscopic world seems to imply a collapse of the wave function. However, accepting this collapse entails conflicts with relativity theory as well as with T invariance, and therefore many interpretations have so far avoided it.

And yet, we have pointed out several indications that these two dismissals are inadequate – that there is more to time than just a dimension, and that the wave function does undergo a unique change upon interacting with the macroscopic world. Moreover, the alleged collapse affects not only the state of the particle at the moment of observation but, sometimes, its earlier history as well, suggesting that an entire segment of spacetime is subject to subtle evolution.

Could it be, then, that the two phenomena – time's passage and wave-function collapse – are not only real, but the latter is the very manifestation of the former? A wave function, after all, is a sum of many equally possible outcomes, while the measurement brings about the realization of one out of them, the others vanishing. Is this not the very difference between future and past? And is collapse not elusive because it creates the elusive 'now'?

Indeed de Broglie (quoted in [42]) paid tribute to Bergson as a philosophical ancestor of QM. Had Bergson had a chance to study QM, de Broglie asserted, he would learn that Nature hesitates at any instant between several choices, and he would reiterate what he has said in *The Creative Mind*: "Time is nothing but this hesitation."

Here, then, is the unfavored hypothesis of Becoming again, now with a cosmological twist. Suppose that there is indeed a 'now' front, on the one side of which there are past events, adding up as the 'now' progresses, while on its other side there are *no* events, and hence, according to Mach, *not even spacetime*. Spacetime thus 'grows' into the future as history unfolds. Time's asymmetry would therefore be naturally anchored in this alleged progress of the 'now'.

Notice that by ascribing to the 'now' the very creation of spacetime itself, we do away with all the logical difficulties which have so far beset the 'moving now' hypothesis (see Sect. 17.2). Our hypothesis is merely an extension of the Big Bang model, taking advantage of its logical rigor: If the Big Bang has created not only matter and energy but also spacetime itself, then no one needs to worry about 'what happened before the Big Bang' or 'what lies outside spacetime'. Similarly for our hypothesis, the 'now' does not move on some pre-existing dimension but rather creates that dimension. This is not 'movement' in the ordinary sense, so no endless series of time parameters is entailed by it.

Now let this Becoming be made quantum mechanical. What role does the wave function play in this creation of new events? The dynamically evolving spacetime allows a radical possibility. Rather than conceiving of some empty spacetime within which the wave function evolves, the reverse may be the case: *The wave function evolves beyond the 'now', i.e., outside of spacetime, and its 'collapse' due to the interaction with other wave functions creates not only the events, but also the spacetime within which they are located in relation to one another*. The famous peculiarities of the quantum interaction – nonlocality, the coexistence of mutually exclusive states, backward causation and the inconsistent histories presented in the previous sections, thus become more natural.

Can the reciprocal effects of spacetime and matter – the celebrated lesson of general relativity – thus possibly gain a quantum mechanical explanation? Perhaps it is the wave function, we submit, that is more primitive than spacetime, and the spacetime connecting two events is the product of their interacting wave functions. We shall close with a more audacious consequence of this hypothesis for quantum field theory. Perhaps the wave function of a force-carrying boson, such as a graviton or photon, which, by our hypothesis, creates also the spacetime within which the final interaction is completed, determines the spatiotemporal distance between the events. In other words, 'attraction' and 'repulsion' may be the consequences of the specific spacetime metric created by the interacting wave functions.

Only the future, be it a fixed Minkowskian one or an open Bergsonian one, will decide whether this sketchy proposal will eventually mature into a viable theory of spacetime dynamics. We can only plead that the questions raised and the odd phenomena pointed out in the preceding pages call for radically new ways of thinking about quantum phenomena and spacetime.

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18 Event-Based Quantum Theory

Geoffrey F. Chew

Despite the process discretization implied by Copenhagen rules for interpreting local quantum theory, time quantization seems to be precluded by relativity. Nevertheless a quantum representation of a redshifting spatially-homogeneous universe may be based on discrete-step Feynman paths carrying Poincaré-invariant action – paths that not only propagate the wave function but provide an elementary particle Hilbert space basis. Any path is an 'event graph', at whose vertices elementary particle arcs commence and terminate. Local path steps are at the Planck scale while, at a much larger 'measurement scale', global steps separate successive wave functions. Wave function spacetime is but a tiny fraction of path spacetime.

Electromagnetic and gravitational actions are 'at a distance' in the Wheeler–Feynman sense, while strong (color) and weak (isospin) actions, as well as kinetic action, are 'local' in a sense paralleling the action of local field theory. Polarization-rotation of particle arcs within event graphs defines a localized energy that not only controls kinetic action but provides a source for gravity analogous to the electric charge source of electromagnetism. Rest mass is generated by self-action within an arc extended by 'zitterbewegung events' that reverse lightlike arc velocity.

Photons coupled to conserved electric charge and gravitons coupled to energy enjoy privileged status among elementary fermions and vector bosons. Although 'measurement' lacks a priori meaning, the phase of the complex wave function facilitates accumulation of 'information' through 'gentle' electromagnetic and gravitational events that emit soft photons and gravitons. Through soft photon–graviton coherent states corresponding to classical fields, the (global) wave function accumulates an 'information reservoir' as the universe expands.

18.1 Introduction

Because the Copenhagen formulation of quantum theory refers explicitly to measurement – which, discretely in time, 'collapses' the wave function – whereas any measurement requires long-range aspects of electromagnetism and gravity, the absence in the Copenhagen rules of any distinction between short- and long-distance interactions is inconsistent. Whitehead's notion of

discrete process [1] – a collection of spacetime-localized events – enables a cosmological reformulation of quantum theory that promises to elucidate the meaning of measurement without according this notion a priori status.

A global quantum theory may be founded on events that include huge numbers of 'gentle' events which emit 'soft' photons and gravitons to build up a cumulative 'information reservoir' within a global wave function through coherent states corresponding to classical fields of extremely low frequency. A gentle event disturbs 'observable matter', i.e., matter other than soft massless bosons, to a degree that is negligible for all practical purposes. A 'Copenhagen wave function' restricted to a 'laboratory' region of spacetime that is tiny on the Hubble scale, and attending only to observable matter, does not recognize the information reservoir.

The (classical) 'event' concept may be incorporated into a discrete, i.e., 'stepped', Feynman (classical) path whose action controls wave function propagation [2]. Path discreteness dovetails with the discreteness of elementary particles (off-shell as well as on-shell) that provide a basis for Hilbert space. Any path may be described as a (closed) 'event graph' – a set of event vertices connected by elementary-particle-associable arcs. Any arc connects a pair of vertices through a succession of alternating 'forward' lightlike and spacelike intervals of spacetime, maintaining a fixed lightlike velocity direction throughout the 'life' of the arc. A vertex of an event graph is a potential event where elementary particles may be created or annihilated. Stationary action within the sum over all possible paths that build a propagator, together with the wave function to be propagated, chooses those events that are 'likely to occur'.

How does an arc distinguish different types of elementary particle? Any stepped path is at once an event graph and a set of directed labeled closed loops, each loop a sequence of 'pre-events' separated by lightlike Planck-scale steps in spacetime. A pair of discrete labels is carried by each pre-event loop. One loop label is 2-valued and the other 3-valued, as detailed in Appendix A. Different loops are correlated within an event graph. Any graph arc comprises a quartet of loop segments whose patterns of labels, detailed in the appendices, match the quantum numbers – electric charge, isospin, color, fermion chirality, and generation – carried by the Standard Model's elementary fermions and vector bosons. The appendices explain how the loop-quartet structure of a fermion arc differs from that of a boson arc. Termination of an arc at an event dissolves the loop quartet and redistributes the arc's four constituent loops. Pre-events build the vertices as well as the arcs of an event graph.

Event-based quantum theory (EQT) displays numerous unusual aspects. The special role of electromagnetism and gravity will not be addressed in this introductory discussion. Two other facets of EQT receive priority. One is discretization of time, and the other is a status for event graphs that is

more fundamental than that of Hilbert space. These two nonstandard aspects of EQT are related.

Discrete classical paths, whose causal Poincaré-invariant actions determine wave function propagation, provide an elementary particle EQT basis for Hilbert space – a basis sketched in the body of this paper and elaborated in the appendices. A basis in fields or strings that vary continuously with local time is not possible. I shall explain how a EQT path-based wave function represents matter at a fixed 'age'. Discretization of wave function age will be seen to be essential to consistent contact between paths and wave function. The model incorporates Whitehead's idea that discrete process is more fundamental than matter by representing process as Feynman-path steps from a pre-event to a subsequent pre-event through an age-discretized 'path spacetime' within which only a minuscule portion is occupied by material-representing wave functions.

In standard quantum theory the wave function changes continuously with time. The action of standard-theory continuous Feynman paths can be represented by classical entities such as fields or strings that carry continuous spacetime labels. Action is spacetime localized in a standard way. Identifying the classical action vehicle as 'material', process in standard particle physics may be said to enjoy material underpinning. As Whitehead appreciated almost a century ago, a converse relationship between matter and process cannot be represented without discretizing process.

Process discretization is feasible in the spacetime proposed by Milne [3] during the 1930s in order to connect Hubble's redshift to a principle of universe spatial homogeneity. Milne attached a different Lorentz frame – a 'local frame' – to each point in a 3-space of definite age. Lorentz invariance represented homogeneity of this space. (Although not appreciated at the time of Milne's work, the phenomenological meaning of the local frame is provided by approximate isotropy of cosmic background radiation in any observer's local frame.) Milne's fourth dimension, which I call 'age', is Lorentz-frame independent and correspondingly may be discretized. Discrete process can be represented by Feynman paths without a priori material association, paths whose loop constituents take discrete steps forward or backward in age between successive pre-events. Pre-events, together with global age steps between successive wave functions related by the path-action-determined Feynman propagator, can provide a Hilbert space basis if and only if the wave function age step is an integral multiple of the loop step.

The (local) path step in age δ is at the Planck scale of $\sim 10^{-43}$ s, while the (global) wave function step Δ is at the considerably greater 'measurement scale'. The wave function step is tentatively guessed to be near 10^{-5} s (in a local frame), above the atomic scale but still below the scale of human consciousness – a step allowing the wave function an S-matrix interpretation. (Brain wave frequencies are $\sim 10^{-1}$ s.) In language introduced long ago by quantum field theory, the local step may be said to provide 'ultraviolet-

divergence cutoff', while the global step provides infrared cutoff. Disregarding frequencies below Δ^{-1} leads Copenhagen quantum physics to associate measurement with 'wave function collapse'.

The huge integer ratio Δ/δ ($\sim 10^{38}$) between local and global steps is presently accepted as a fundamental EQT parameter, enjoying a status like the scale interval spanned by inflation in standard cosmology. The first wave function of the universe is at the age of a single global step. Eventually I hope number theory will point to a precise prime-integer value for the ratio between wave function step and path step. An example of how number theory can pick out a special huge prime is provided by the Mersenne prime sequence, $2^2-1=3$, $2^3-1=7$, $2^7-1=127$, $2^{127}-1$, brought to my attention by P. Noyes and H. Doughty.

18.2 Age Discretization

Milne's spacetime occupied the interior of a forward light cone. The EQT 'path spacetime' factorizes a forward light cone interior into the product of a curved 3-dimensional 'boost space' and a 1-dimensional 'age space'. Age – the Minkowski distance from the light cone vertex – is Lorentz invariant and may be quantized compatibly with Milne's principle of equivalence for all locations in 3-space. EQT applies Milne's homogeneity (expressed by Lorentz invariance) principle to the action-carrying process that underpins wave function propagation. Using the symbol τ for age and the 3-vector symbol $\boldsymbol{\beta}$ for a (dimensionless) 'boost', infinitesimal displacements in ordinary 3-space and in boost space are related by $\mathrm{d}\boldsymbol{x} = \tau \mathrm{d}\boldsymbol{\beta}$. Writing the boost symbol as $\beta\boldsymbol{u}$, where $\boldsymbol{u}$ is a unit 3-vector, the curved metric of Milne's continuous boost space is $\mathrm{d}\boldsymbol{\beta}^2 = \mathrm{d}\beta^2 + \sinh^2\beta\,\mathrm{d}\boldsymbol{u}^2$.

'Path spacetime' – the habitat of pre-events – is the set of hyperboloids whose age is a positive integer multiple of the EQT Planck scale unit δ. Starting from one point on a hyperboloid, any other point on this hyperboloid may be reached by a boost. Choice of frame amounts to choosing, at specified age, a spatial location, i.e., to selecting a point in continuous boost space.

The light cone boundary of Milne's spacetime corresponds to a 'big bang' from which age is measured. An 'arrow of time' is implicit; redshift and universe homogeneity are correlated. However, when Milne supposed age continuity and interpreted his continuous spacetime materialistically (rather than through process), he encountered vanishing 4-dimensional curvature, incompatible with Einstein's classical general relativistic representation of gravity. Milne's classical materialistic approach, while widely recognized as interesting, was judged unviable. (It came to be called kinematic cosmology.)

The age discreteness of EQT renders spacetime curvature undefinable. Gravity is represented a priori not by spacetime curvature but through event-graph Feynman paths by the action-at-a-distance concept uncovered

by Wheeler and Feynman in their 1949 representation of classical electromagnetism without fields [4]. Each event graph carries a symmetric Lorentz-traceless gravitational second-rank tensor potential whose discrete sources are the individual forward arc steps within the graph. The gravitational constant is represented in EQT through the lightlike path step in age that defines a 'Planck unit of energy'. (The gravitational constant is of order δ^2 in units where $\hbar = c = 1$.) The gravitational action of a forward arc step is the bilinear inner product of its spacetime-displacement 4-vector and energy–momentum 4-vector (both lightlike and parallel to each other) with the gravitational tensor potential at the step location. The large-scale meaning of classical sub-lightlike material trajectories derives from stationary phase within wave function propagation. EQT sub-lightlike velocity originates in quantum fluctuation of lightlike velocity – zitterbewegung – as discussed below.

Whereas path spacetime is the set of hyperboloids whose age is a positive integer multiple of the path age step δ, wave function spacetime is the relatively tiny subset of these hyperboloids whose age is a multiple of the 'global step' Δ, the ratio Δ/δ being a huge integer ($2^{127} - 1$?). The integer ratio populates wave function spacetime by particle-interpretable quartets of pre-events that allow Feynman paths to contact Hilbert space in an elementary particle 'coordinate basis'. A conjugate 'momentum–spin basis' is defined by a norm-preserving Gelfand–Naimark (not Fourier) transform [5] that cannot be discussed here.

18.3 Action

EQT path action, which determines wave function propagation, includes three (separately Poincaré-invariant) components carried by event-graph arcs (as opposed to action components carried by graph vertices). Gravitational and electromagnetic actions-at-a-distance, carried by each particle-arc step, are proportional to arc energy and arc electric charge, respectively. Arc energy is proportional to the mean rate of loop-quartet rotation around the lightlike straight line in spacetime that is followed stepwise, in a forward arc segment, by the quartet's 'center'. (The local frame spatial distance from the quartet center to any non-centrally located quartet pre-event is $c\delta$.) Quartet rotation is characterized as that of an arc's 'transverse polarization'. A unit spacelike polarization 4-vector attaches to the start and finish of any arc step. The direction of the polarization rotation in an arc step, left or right, determines (lightlike) arc helicity. Because the rotation occurs in discrete angular steps (of fixed magnitude as well as direction for any arc), the maximum rate $2\pi/\delta$ implies a maximum energy (in the local frame) of the order of c^2 times the so-called Planck mass. The third arc-carried action component is kinetic action, proportional to arc energy.

Weak (isospin) and strong (color) local components of action are carried by certain event-graph vertices. At most of these vertices, pre-event loops

are directed in a Chan–Paton pattern [6] that allows the EQT action prescription to imitate the gauge-invariant CP-invariant Yang–Mills action [7] of the Standard Model. Vertex action is generally proportional to Lorentz-invariant inner products of the lightlike 4-velocities and transverse polarization 4-vectors belonging to the arcs terminating at the vertex. CP transformation of an EQT Feynman path reverses all loop directions. (Although the pre-event loop structure of all arcs and most vertices is symmetric with respect to loop-direction reversal, vertices connecting a W arc to a pair of quark arcs are asymmetric.)

18.4 Path Zitterbewegung

According to EQT, a 'primitive' rest mass arises dynamically from actionless 'zitterbewegung' (zbw) events that reverse the lightlike velocity direction of an arc while changing neither arc energy nor helicity nor electric charge (nor color nor generation). Although self-action-at-a-distance is absent from an individual arc (of single lightlike velocity), a succession of arcs connected by zbw events carries gravitational and (if electrically charged) electromagnetic action-at-a-distance between causally connected opposite-velocity forward arc steps of the succession. In the absence of any action other than kinetic and self-action-at-a-distance, a single elementary-particle wave function of definite momentum–spin that is some special superposition of two opposite lightlike velocities – one parallel to the momentum and one antiparallel – can be a 'stationary' state. The ratio within the wave function of the amplitudes for these opposite velocities determines an average sub-lightlike velocity which, together with momentum magnitude, implies a 'primitive' rest mass.

Because the Hilbert space is a Fock space, the wave function of an 'observable' particle includes products of several elementary-particle wave functions. Measurable rest mass reflects multiparticle elementary off-shell constituents of a single observable on-shell particle – roughly the equivalent of 'mass renormalization' in standard quantum field theory [7].

A familiar rest-mass meaning related to primitive rest mass is implicit in Dirac's equation for a single electron, once Dirac's lightlike electron-velocity operator, which commutes with the electron-location operator, is recognized [8]. An ad hoc phenomenologically-motivated term in Dirac's Hamiltonian, which fails to commute with the velocity operator, reverses the lightlike velocity and at the same time supplies rest-mass-generating action beyond the kinetic action of lightlike motion. The term 'zitterbewegung' was coined by Schrödinger to describe the Dirac lightlike-velocity fluctuation. Note that, in contrast to the Dirac equation, the EQT representation of primitive rest mass requires no extra arbitrary parameter. (Dirac failed in his attempt to interpret a solution of his equation as a single-electron wave function because finite-dimensional representations of the Lorentz group are not unitary. EQT

employs the infinite-dimensional unitary representations found by Gelfand and Naimark [5], which depict single elementary particles that may be off-shell in both momentum and spin [9].)

In order to preserve the distinction between long- and short-range interaction, an EQT path constraint forbids graviton or photon zbw events. At present no reason is seen to exclude gluon zbw. Gluons as well as neutrinos are expected to exhibit a gravitationally-generated 'primitive' mass matrix. The primitive rest masses of electrically-charged elementary particles are plausibly dominated by electromagnetic self-action.

Beyond the scope of this paper is the difference between fermion and boson zitterbewegung related to a 'chirality' carried by fermions but not by bosons and to the difference between boson and fermion quartet structures (see appendices). We remark only that fermion-zbw chirality reversal involves fixed-age (spacelike) tachyon arcs that do not cross wave-function hyperboloids and that correspondingly are not represented by the Hilbert space. (Tachyon arcs carry neither energy-electric-charge nor color-generation.) Not yet understood is the consequence of long-distance tachyon connection, within an event graph, between different fermion-arc zbw successions.

18.5 Conclusion

I have sketched a quantum cosmology designed to span all ages after the Big Bang and all scales between that of Planck and that of Hubble. The wave function, representing matter, has its basis in process. Undiscussed has been the arbitrariness residing in the initial wave function at age Δ. The 'size' of the material universe has not been considered here, nor has the influence of initial conditions on universe 'self-knowability' and capacity for 'creativity'. The proposal, however, accords gravity and electromagnetism a distinguished status intended to enable universe self-awareness.

EQT requires four ingredients:

- A discretized path spacetime that includes a (much smaller) wave function spacetime, allowing paths to provide the basis for Hilbert space.
- Path constraints that allow an event graph to contact a wave function.
- Specification of the event-graph action that determines wave function propagation.
- The wave function initial ('Big Bang') condition.

This report has touched on the first three of these ingredients.

EQT lifts a notion from Feynman graphs in order to connect wave functions to paths. Feynman paths and Feynman graphs are correctly regarded as completely different, the latter being tied to a perturbation approximation while the former is a general concept. Feynman graphs, however, by uncovering the idea that 'positrons are like electrons moving backward in time', led Feynman to observe that a single 'curve' meandering backward

and forward in time, as well as in space, might represent all the electrons and positrons in the universe. EQT seizes on Feynman's insight to find in individual classical closed event graphs a basis for the entire universe which, with zitterbewegung, creates primitive rest mass and offers a quantum explanation of general relativity's classical spacetime curvature.

EQT replaces Copenhagen 'laboratory wave function collapse' by global wave function evolution of coherent states that 'record the past', in the sense that a coherent soft (i.e., beyond measurement scale) off-shell graviton–photon state is a classical field which reflects its material sources. Despite the resemblance with the Standard Model in its Hilbert space (an elementary particle Fock space) and kinetic plus event-carried (local) action, the EQT paths whose actions determine wave function propagation do more than locally propagate and collide particles. Here are some examples of nonlocal (holistic) EQT features:

- Gravitational action-at-a-distance, including future as well as past 'sources' (within process, not within matter) provides global influences that preclude matter isolation even though permitting identification of 'separate pieces of matter'.
- Fermion zitterbewegung involves tachyon arcs that 'instantaneously' correlate pieces of matter with arbitrarily large spatial separation.
- The absence of local action in a gravitational or electromagnetic event (where a graviton or photon arc terminates) allows such events to be 'gentle', disturbing observable matter only slightly while nevertheless 'recording' the event in EQT's coherent state 'information reservoir'. The reservoir classical field has frequencies below the measurement scale set by the wave function global age step. This thereby enables the miracle of information accumulated with negligible material disturbance.

An essential future step for EQT is the development of approximation strategies that will allow the successes of continuous-time theories to be reproduced. Only the outlines of such strategies, based on the large ratios of Hubble scale to measurement scale and of the latter to Planck scale, have so far been conceived. In this respect an effort is underway to quantify the accuracy of 'observation reproducibility' – the foundation of science even though exact reproducibility of any measurement is impossible in an expanding universe. Through special relativity, Einstein enlarged the meaning of reproducible observation from observations related by the spacetime displacement group to those related by the so-called Poincaré group. Understanding the accuracy of special relativity in EQT may require some Mach-like meaning for 'isolated observable matter' defined by a 'sea of process'. Also required, of course, is a meaning for measurement.

The huge ratios explicitly recognized by EQT reduce a variety of hitherto sacrosanct ideas (such as measurement reproducibility) to the status of extremely accurate approximations. Do any sacred principles of physics survive? Undisturbed are electric charge and angular momentum conservation

and Pauli's symmetry that defines 'identical particles', a notion evidently prerequisite to any meaning for reproducible measurement.

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Henry Stapp and Jerry Finkelstein have greatly influenced the developments reported here.

18.6 Appendixes

Quartets of pre-events provide an elementary particle basis for EQT Hilbert space. A wave function pre-event is labeled continuously by its spatial location and discretely by a 2-valued 'pre-event spin' (e-spin) index plus 3-valued 'electro' index – labels that, with symmetry constraints on a quartet's internal wave function, define isospin, color, generation, and chirality in a manner compatible with boson–fermion distinction.

Internal quartet wave functions distinguish gravitons, electroweak vector bosons (30° elementary Weinberg angle), gluons, GUT bosons and 3 generations of quarks and leptons. Fermion chirality reversal combines loop-direction inversion with 'conjugation' of e-spin and electro labels. Although charged lepton generations are distinguished by conserved e-spin, two of the 3 neutrino generations may mix and third-generation neutrino may mix with antineutrino. Elementary quark generations match lepton generations, but e-spin entangles color with quark generation in quark–W coupling.

Appendix A: Pre-Event Labels and Associated Particle Quantum Numbers

Denoting by the symbols $+$, 0, $-$ the three alternative values of the electro label on a path loop and by the symbols $\uparrow$, $\downarrow$ the two values of the e-spin label, a 6-valued index denoted p designates the index pair attached to a loop. To each different value of p a different integer may be associated so that conjugation, defined as the set of label interchanges, $+ \leftrightarrow -$, $0 \leftrightarrow 0$, $\uparrow\leftrightarrow\downarrow$, changes the sign of p. (Conjugation resembles the operation called charge conjugation in standard particle physics.) We employ the 6 integers ±1, ±2, ±3 such that

$$\begin{array}{llllll} p=+1 & \text{for } +\uparrow, & p=+2 & \text{for } -\uparrow, & p=+3 & \text{for } 0\uparrow, \\ p=-1 & \text{for } -\downarrow, & p=-2 & \text{for } +\downarrow, & p=-3 & \text{for } 0\downarrow\ . \end{array} \tag{18.1}$$

Continuity of p throughout any path loop yields a separate conservation rule for each of the 6 different p labels. According to Feynman's graph principle, a conserved quantity proceeding backward in age counts as the negative of this quantity when moving forward. Adapted to a discrete directed age loop,

Feynman's rule attaches to a pre-event, along a loop of index p, a quantity $N_p = +1$ if the preceding pre-event occurs at an earlier age and the following pre-event at a later age. In the converse situation, $N_p = -1$. To an age-turnaround pre-event whose preceding and following pre-events share the same age, the assignment is $N_p = 0$. (A pre-event belonging to a loop with index p is assigned $N_{p'} = 0$ for $p' \neq p$.) For any path, summation over all pre-events with equal age then yields zero at any age for each of 6 total-N_p values.

Any linear combination of the 6 conserved N_p values is also conserved. It is useful to define the 6 linearly-independent combinations

$$N^{\pm}_{|p|} \equiv N_{|p|} \pm N_{-|p|} \ , \qquad |p| = 1, 2, 3 \ , \tag{18.2}$$

because the requirement that tachyon arcs be 'self-conjugate', together with N_p conservation in any event for all 6 values of p, then means that any $N^-_{|p|}$ is conserved in individual events by the participating particle arcs. An $N^+_{|p|}$ (transportable by tachyon as well as by particle arcs) is only globally conserved by particle arcs.

The two 'observable-via-particles' locally-conserved quantum numbers emphasized in what follows – electric charge and e-spin, superselected by elementary particle wave functions – are

$$Q \equiv \frac{1}{3}(N^-_1 - N^-_2) \ , \qquad e \equiv \frac{1}{2}(N^-_1 + N^-_2 + N^-_3) \ . \tag{18.3}$$

Any function of the 3 quantum numbers $N^-_{|p|}$ that is not determined by Q and e fails to be superselected even though locally conserved. We identify Q with elementary particle electric charge in units of charge carried by an elementary positron. The quantity e will relate to color and generation.

The quantum number

$$M \equiv N^+_1 + N^+_2 + N^+_3 \ , \tag{18.4}$$

which takes the value $+1$ for any pre-event within an age-advancing loop segment and the value -1 within any age-retreating loop segment, has zero aggregate value for any arc. Although M is invisible within Hilbert space, the internal structure that distinguishes elementary particle types, as well as the structure of events, depends on M.

Appendix B: Arc Structure

An arc combines two age-advancing path segments with two age-retreating segments in a 'rigid body' configuration whose 'center' (defined below), in forward arc steps as age increases, follows an age-independent lightlike direction in spacetime. Because arc crossing of a wave function hyperboloid occurs between two successive forward steps, EQT Hilbert space is built from forward arc-step quartets of common-age pre-events.

Two distinct spatial structures allow quartets to be classified as either 'boson' or 'fermion'. In the local frame of a boson quartet center, the direction of the displacement between the two age-advancing pre-events is orthogonal to the displacement between the two age-retreating pre-events, both these 'transverse' displacements being orthogonal to the 'longitudinal' direction followed by the arc's center. The distance from either a boson or a fermion quartet center of any pre-event not located at the center is $c\delta$. In a fermion quartet either the ($M = +2$) age-advancing pre-event pair or the ($M = -2$) age-retreating pair are located at the quartet center. We characterize the coincident pre-event pair at the arc center as 'pinched'.

Quartets of loop labels are constrained. Uniquely among particle arcs, graviton arcs have all 4 electrolabels 0. To describe general constraints, we denote by the term 'arc half' the pair of age-advancing (positive M) loop segments and by 'antihalf' the age-retreating (negative M) pair. Because any tachyon-arc label pair is self-conjugate, tachyon-half electro labels (half or antihalf) are 00 or $+-$ while e-spin labels are $\uparrow\downarrow$. For non-graviton particle arcs, one electro-label constraint requires both half and antihalf electro-label pairs to be within the label-pair subset ($++$, -0, $+-$) or within the conjugate subset ($--$, $+0$, $-+$), allowing a total of only 18 different particle electro labelings. Further constraints on a pinched half or antihalf of a fermionic particle arc exclude therefrom the $+-$ label pair under all circumstances and the 0 label unless the accompanying (unpinched) antihalf or half carries the $+-$ label pair. The foregoing electro-label particle-arc constraints relate to the feature that photons couple to electric charge, described in Appendix G. E-spin labels on particle arcs are unconstrained apart from the general requirement that the two 6-valued labels carried by any unpinched pre-event pair in any arc must differ.

Appendix C: Remarks about EQT Hilbert Space

EQT Hilbert space is based on those pre-event quartets located in wave function spacetime – a minuscule fraction of all pre-events along a path. Six continuous ('Lorentz-space') parameters, that locate a quartet center on its wave-function hyperboloid and orient the quartet about its center, define an infinite-dimensional single-particle 'external' Hilbert space discussed in [9]. The space is constrained by a collection of Pauli-like symmetries, the unfamiliar symmetries being detailed in Appendix E. The usual symmetries make the EQT Hilbert space into a Fock space. Labels on the 4 pre-events of a quartet (from several different path loops) spawn particle quantum numbers, the two distinct arc structures described in Appendix B allowing quartets to be classified either as 'boson' or as 'fermion'.

These appendices, via symmetry constraints and superselection rules for a finite-dimensional 'internal' quartet Hilbert space, define elementary-particle 'types' through label structure. The 2-valued e-spin label spawns color and generation, while the 3-valued electro-label generates isospin as well as electric

charge. The finite number of distinct particle types will be seen to match, apart from elementary scalars, a GUT extension of the Standard Model.

Appendix D: Quartet Internal Wave Function

Appendix A introduced the symbol p for the 6-valued pre-event label pair. The 'internal' basis states available to represent an elementary particle, before symmetry constraints are imposed, may be denoted algebraically by

$$\left\{p_1', p_2'\right| \cdot \left|p_1, p_2\right\} ,$$

where labels within the bracket $|\ \}$ are those of $M = +1$ (age-advancing) pre-events, while labels within the antibracket $\{\ |$ refer to $M = -1$ (age-retreating) pre-events. (The foregoing direct-product notation must not be confused with Dirac's bras and kets.)

The order of the two labels within a bracket is given meaning for an unpinched half or antihalf by the different spatial locations of the two pre-events. As indicated earlier, these appendices ignore 6 continuous external parameters that spatially locate and orient the quartet. For a fixed set of pre-event spatial locations, there are 2 possible permutations of the positive M or negative M labels. Because any unpinched label pair must differ, label transposition in an unpinched bracket leads to a different quartet of the same 'external' spatial orientation. A symmetry to be imposed makes it unnecessary to be more specific. In a pinched bracket, label order is meaningless.

Using a single label h for the (ordered) label pair belonging to an entire half or antihalf, the internal single-particle wave function aspect of the M-reversing operation conventionally designated in particle physics as CP is

$$\left\{h'\right| \cdot \left|h\right\} \longleftrightarrow \left\{h\right| \cdot \left|h'\right\} . \tag{18.5}$$

We shall refer to the wave function operation (18.5) as inversion. (The path meaning of inversion, applicable to both particle and tachyonic arcs, is reversal of loop directions while maintaining pre-event labels and locations in path spacetime. Event structure is not always inversion symmetric [10].)

The EQT meaning of chirality involves both inversion and the label conjugation defined above in Appendix A. A zbw event generating fermionic primitive rest mass (by velocity reversal) combines inversion with label conjugation [10]. For the photon wave function, we shall find that label conjugation corresponds to the operation standardly called charge conjugation. But generally the adjective 'charge' may be misleading because e-spin is reversed as well as electro label. Generation and color are involved.

There will be frequent reference in these appendices to label conjugation. The effect of conjugation on a half label will be denoted $h \leftrightarrow h^*$ $(h^{**} = h)$. Elementary fermion chirality reversal combines conjugation with inversion in an operation

$$\left\{h'|\ \cdot\ |h\right\} \longleftrightarrow \left\{h^*|\ \cdot\ |h'^*\right\}, \tag{18.6}$$

which leaves the two locally-conserved elementary-particle 'quantum numbers' Q and e unchanged.

Appendix E: Quantum Superposition of Quartets. Quartet–Half Symmetry Constraint

Subject to Q, e superselection and maintenance of the boson–fermion distinction, internally different quartet wave functions are superposable. Elementary particles belong to special symmetrical superpositions of different quartet patterns that share the same locally-conserved quantum numbers Q and e. Symmetry of quartet internal superposition segregates EQT elementary particles into 'families'.

EQT imposes a 'Pauli-type' constraint on half and antihalf wave functions in the quartet internal Hilbert space. Although these appendices ignore the external wave function, double covering of the Lorentz group is rendered consistent with classical EQT paths by the 2 permutations of the 'oppositely'-located labels on an unpinched half or antihalf. Distinguishability of these 2 labels permits a requirement that any unpinched half or antihalf wave function be antisymmetric under label interchange. In contrast, a pinched half wave function (where label permutation is meaningless) is label-symmetric. (Pre-event label interchange within either half or antihalf of a quartet not only leaves Q and e unchanged but fails to alter path action.) Internal symmetry is a Hilbert space constraint supplementing those constraints inherited from an arc's classical structure. The external wave function exhibits the usual Pauli symmetry (antisymmetry) that defines 'identical particles'.

Tailored to Yang–Mills couplings [7] is the matrix character of the half–antihalf product basis for the internal Hilbert space that is exhibited by the full-state direct-product notation $\left\{\phi(h)|\ \cdot\ |\psi(h)\right\}$. The antihalf wave function ϕ and the half wave function ψ individually exhibit either symmetric (pinched) or antisymmetric (unpinched) dependence on a pair of p labels. Denoting half symmetry (antisymmetry) by a plus (minus) sign, all elementary vector-boson internal wave functions exhibit the symmetry structure $\left\{-|\ \cdot\ |-\right\}$, while elementary fermions and antifermions enjoy the symmetry structures $\left\{-|\ \cdot\ |+\right\}$ or $\left\{+|\ \cdot\ |-\right\}$.

We reduce the internal Hilbert space by requiring factorization of electro and e-spin half and antihalf subspaces, with electro and e-spin basis states that are separately symmetric or antisymmetric under transposition. The 2-valuedness of e-spin leads to the familiar antisymmetric singlet and symmetric triplet of half states, displayed below in (18.11) and (18.12). Despite the 3-valuedness of the electro label, the label constraints of Appendix B allow (18.8)–(18.10) to define an antisymmetric singlet of zero (net) electric charge and a pair of symmetric doublets each of whose 4 members carries nonzero electric charge.

Elementary particle internal factorization of electro and e-spin quantum correlations casts half states into 3 categories denotable by the $M = +2$ symbols

$$|-+\}\,, \quad |+-\}\,, \quad |++\}\,, \tag{18.7}$$

the first two of these categories being unpinched, while the third is pinched. The left index in (18.7) denotes electro label symmetry and the right index e-spin symmetry. Inversion generates a matching set of $M = -2$ antihalf categories. (Our antihalf convention is that the left index gives e-spin symmetry while the right index gives electro symmetry.) Absence of spatial extension in a pinched half or antihalf precludes wave functions with $--$ symmetry structure.

The boson sectors have symmetry structures $\{+-|\cdot|-+\}$, $\{-+|\cdot|+-\}$, and $\{+-|\cdot|+-\}$, plus the latter's inverse, while fermion structures are $\{++|\cdot|-+\}$, and $\{++|\cdot|+-\}$, plus inverses. As summarized by Table 18.1, whose content will be gradually unpacked as these appendices unfold, the foregoing sectors embrace, respectively, gluons, gravitons plus electroweak vector bosons, GUT bosons, quarks and leptons (the fermions being of either L or R chirality). E-spin will be found manifested as color in an unpinched half or antihalf with electrolabels $+-$ and as generation in any pinched half or antihalf. An unpinched half with electrolabels other than $+-$ carries neither color nor generation.

The foregoing notation displays parallelism between unpinched fermion halves and vector boson halves. According to Table 18.1, the unpinched half of a quark or antiquark has the structure of a gluon half, while the unpinched half of a lepton or antilepton shares the structure of an electroweak boson half. One half of a GUT vector boson matches an unpinched lepton half; the other matches an unpinched quark half. Half matching enables the Chan–Paton structure of events with Yang–Mills action [6, 7].

Appendix F: Half and Antihalf Electro and E-spin Multiplets

Factoring electro space from e-spin space, the unique self-conjugate antisymmetric ($M = +2$) electro half state is

$$|0\} \equiv \left|(1/2)^{1/2}\left[(+-)-(-+)\right]\right\}\,, \qquad Q = 0\,, \tag{18.8}$$

with $|0^*\} = -|0\}$. [In the present section symbols such as $(+-)$ and $(-+)$ describe electro labels, not the half symmetry of Appendix E.] The ($M = -2$) inverse of (18.8) is denoted $\{0|$. We shall find (18.8) to behave, in standard parlance, as an isosinglet.

The electro-doublet symbol $|2\}$ means the pair of charge-symmetric half states

Table 18.1. Standard Model sectors

Sector	Quartet symmetry	Q	e
Vector boson			
Gluon	$\{+-\vert\cdot\vert-+\}$	0	$2,1,0,-1,-2$
Electroweak	$\{-+\vert\cdot\vert+-\}$	$1,0,0,-1$	0
GUT antiboson	$\{-+\vert\cdot\vert-+\}$	$-2/3,+1/3$	$+1,0,-1$
GUT boson	$\{+-\vert\cdot\vert+-\}$	$+2/3,-1/3$	$-1,0,+1$
Fermion			
Quark	$\{+-\vert\cdot\vert++\}$ $*\{++\vert\cdot\vert-+\}*$	$+2/3,-1/3$	$2,1,0,-1,-2$
Antiquark	$\{++\vert\cdot\vert-+\}$ $*\{+-\vert\cdot\vert++\}*$	$-2/3,+1/3$	$-2,-1,0,1,2$
Lepton	$\{++\vert\cdot\vert+-\}$ $*\{-+\vert\cdot\vert++\}*$	$0,-1$	$-1,0,+1$
Antilepton	$\{-+\vert\cdot\vert++\}$ $*\{++\vert\cdot\vert+-\}*$	$0,+1$	$+1,0,-1$

$$\begin{aligned} \vert\Uparrow\rangle &\equiv \vert++\rangle\,, \quad Q=+2/3\,, \\ \vert\Downarrow\rangle &\equiv \big\vert(1/2)^{1/2}\big[(-0)+(0-)\big]\big\rangle\,, \quad Q=-1/3\,. \end{aligned} \tag{18.9}$$

The conjugate doublet $\vert 2^*\rangle$ is

$$\begin{aligned} \vert\Uparrow^*\rangle &\equiv \vert--\rangle\,, \quad Q=-2/3\,, \\ \vert\Downarrow^*\rangle &\equiv \big\vert(1/2)^{1/2}\big[(+0)+(0+)\big]\big\rangle\,, \quad Q=+1/3\,. \end{aligned} \tag{18.10}$$

Although electric charges of the $M=+2$ doublet $\vert 2\rangle$ are the same as for the $M=-2$ conjugate antidoublet $\langle 2^*\vert$, these half- and antihalf-doublets are distinct. Associating chirality reversal with (18.6) implies that if leptons and antileptons of L chirality carry the unstarred doublet, then those of R chirality carry the starred doublet (Table 18.2).

In either half or antihalf, an electrodoublet provides a basis for an $SU(2)$ irreducible representation. The meaning for $SU(2)_{\rm L}$ symmetry is discussed below.

The $e=0$ transposition-antisymmetric self-conjugate e-spin singlet half state is

$$\vert 1\rangle \equiv \big\vert(1/2)^{1/2}\big[\uparrow\downarrow-\downarrow\uparrow\big]\big\rangle\,, \tag{18.11}$$

while the symmetric e-spin triplet $\vert 3\rangle_e$ is defined to be the set of states

Table 18.2. Elementary lepton wave functions. Halves with label 3 or 3^* are pinched

Q	Wave function	Elementary leptons
0	$\{3,\Uparrow\mid\cdot\mid\Uparrow,1\}$	$(\nu_e,\nu_\mu,\nu_\tau)_L$
-1	$\{3,\Uparrow\mid\cdot\mid\Downarrow,1\}$	$(e^-,\mu^-,\tau^-)_L$
0	$\{1,\Uparrow\mid\cdot\mid\Uparrow,3\}$	$(\bar\nu_e,\bar\nu_\mu,\bar\nu_\tau)_L$
$+1$	$\{1,\Downarrow\mid\cdot\mid\Uparrow,3\}$	$(e^+,\mu^+,\tau^+)_L$
0	$\{1^*,\Uparrow^*\mid\cdot\mid\Uparrow^*,3^*\}$	$(\nu_e,\nu_\mu,\nu_\tau)_R$
-1	$\{1^*,\Downarrow^*\mid\cdot\mid\Uparrow^*,3^*\}$	$(e^-,\mu^-,\tau^-)_R$
0	$\{3^*,\Uparrow^*\mid\cdot\mid\Uparrow^*,1^*\}$	$(\bar\nu_e,\bar\nu_\mu,\bar\nu_\tau)_R$
$+1$	$\{3^*,\Uparrow^*\mid\cdot\mid\Downarrow^*,1^*\}$	$(e^+,\mu^+,\tau^+)_R$

$$|3\}_{+1}\equiv|\uparrow\uparrow\},\quad |3\}_0\equiv\left|(1/2)^{1/2}[\uparrow\downarrow+\downarrow\uparrow]\right\},\quad |3\}_{-1}\equiv|\downarrow\downarrow\}. \tag{18.12}$$

Notice that, under conjugation, $|1^*\} = -|1\}$ while $|3^*\}_e = |3\}_{-e}$. Inversion defines matching antihalf states $\{1|$, ${}_e\{3|$ and their conjugates.

Appendix G: Standard Families of Elementary Particles

Electroweak Bosons

Using the notation of Table 18.1, the electro label particle-arc constraints of Appendix B allow $\{-+|\cdot|+-\}$ $e=0$ quartet wave functions, $\{1,2|\cdot|2,1\}$ or $\{1^*,2^*|\cdot|2^*,1^*\}$. Members of this family have Q values $+1,0,0,-1$. Four superpositions will be found to match the Standard Model's electroweak vector boson sector. Throughout the present section, for notational economy, we factor out and suppress designation of the e-spin-singlet wave function $\{1|\cdot|1\} = \{1^*|\cdot|1^*\}$. (Electroweak vector bosons carry neither color nor generation.) A graviton has the same e-spin structure as an electroweak boson but the value 0 for all four of its electro labels – not the product of the antisymmetric electro label half wave functions $\{0|\cdot|0\}$ defined by (18.8).

Central to EQT Hilbert space, because electric charge is central, is the $Q=0$ elementary photon wave function

$$\text{photon}_{\,\text{elem}} = \left(\frac{1}{2}\right)^{1/2}\{\Uparrow|\cdot|\Uparrow\} - \left(\frac{1}{2}\right)^{1/2}\{\Uparrow^*|\cdot|\Uparrow^*\}. \tag{18.13}$$

Any photon arc (beginning and ending in an electromagnetic event) may contact the wave function (18.13). The special structure of (18.13) – absence of electro label 0, inversion symmetry, conjugation antisymmetry and e-spin-singlet separately in half and antihalf – allows photon coupling to electric

charge, wherever located [10]. This feature we presume (leaning on gauge field theory [7]) will maintain zero rest mass for physical photons when interactions between elementary particles are considered.

Standard Model charged members of the electroweak family distinguish, through their couplings, between lepton chiral labels L and R – the charged vector bosons coupling only to L chirality (parity violation). Although EQT must eventually explain the correlation between chirality, helicity and electro labels, this appendix will be limited to duplicating the content of the Standard Model. Supposing (see Table 18.2) the unpinched halves of standard L leptons to be the electrodoublet $|2\}$, the corresponding elementary charged vector bosons have the CP inverse isospin wave functions

$$\begin{aligned} W^{+}_{\text{elem}} &= \{\Downarrow|\cdot|\Uparrow\}\,, \qquad Q=+1\,,\\ W^{-}_{\text{elem}} &= \{\Uparrow|\cdot|\Downarrow\}\,, \qquad Q=-1\,. \end{aligned} \tag{18.14}$$

An $SU(2)_{\text{L}}$ isotriplet of vector bosons is completed by

$$W^{0}_{\text{elem}} = \left(\frac{1}{2}\right)^{1/2}\{\Uparrow|\cdot|\Uparrow\} - \left(\frac{1}{2}\right)^{1/2}\{\Downarrow|\cdot|\Downarrow\}\,, \qquad Q=0\,, \tag{18.15}$$

but W^{0}_{elem} is not orthogonal to the elementary photon (18.13). The $Q=0$ state,

$$Z^{0}_{\text{elem}} = \left(\frac{1}{6}\right)^{1/2}\{\Uparrow|\cdot|\Uparrow\} - \left(\frac{2}{3}\right)^{1/2}\{\Downarrow|\cdot|\Downarrow\} + \left(\frac{1}{6}\right)^{1/2}\{\Uparrow^{*}|\cdot|\Uparrow^{*}\}\,, \tag{18.16}$$

orthogonal to the photon and such that

$$W^{0}_{\text{elem}} = \left(\frac{3}{4}\right)^{1/2} Z^{0}_{\text{elem}} + \frac{1}{2}\text{photon}_{\,\text{elem}}\,, \tag{18.17}$$

(30° elementary Weinberg angle) is the elementary EQT counterpart of the Standard Model neutral weak vector boson.

Conjugating labels in the foregoing formulas shows inconsistency of a triplet of charged elementary bosons that couple to the (Table 18.2) conjugate 2^{*} doublet of R leptons. Although the charged W and its 'conjugate' would be orthogonal to each other, the state (18.16) and its conjugate are seen not to be mutually orthogonal. The photon wave function (18.13) admits only one (weak) vector boson isotriplet.

What is $SU(2)_{\text{L}}$ symmetry? The 'halved' structure of EQT elementary particles allows the Chan–Paton event pattern [6,7], which leads to $SU(N)$ symmetry of localized action when each half of each event-involved arc carries one of N action-equivalent values of some index. Although the two half labels $\Uparrow$ and $\Downarrow$ are inequivalent to the extent of carrying different quantities of electric charge, EQT electromagnetic path action is nonlocal. Weak path action,

except for quark–W coupling, is localized in $SU(2)_{\mathrm{L}}$-symmetric Chan–Paton events [10]. Many aspects of Glashow–Weinberg–Salam theory [7] promise to survive, although not the idea of an exact $SU(2)_{\mathrm{L}}$ symmetry that is spontaneously broken via elementary scalars. Zero-helicity elementary particles are neither present in EQT nor needed for symmetry breaking.

Gluons

A $Q = 0$ isosinglet $\{+-|\ \cdot\ |-+\}$ family of elementary vector bosons associates with the quartet states ${}_{e'}\{3,0|\ \cdot\ |0,3\}_e$ that Chan–Paton couple to each other and to unpinched quark halves. Because the reflection (18.6) merely permutes common e members within this family of vector bosons, there is coupling to both quark chiralities. Although electric charge vanishes, the 9-member family displays e values $2, 1, 1, 0, 0, 0, -1, -1, -2$.

A symmetric superposition of the three zero-e states (a superposition unaltered either by conjugation or inversion) emerges as 'special' once it is appreciated that e-spin triplet $\otimes$ antitriplet wave functions correspond to a 3×3 matrix. The special state corresponds to the matrix trace. The group $SU(3)$ is isomorphic to the group of unitary unimodular 3×3 matrices, group generators comprising an octet of traceless Hermitian matrices. Because QCD gluons transform in parallel with the generator octet, splitting from the EQT e-spin nonet, a singlet corresponding to the matrix trace allows an orthogonal octet of traceless e-spin states to associate with standard gluons.

We may regard the $|0,3\}_e$ e-spin triplet of 'half' states and the inverse antitriplet as representing 3 'colors' ($e = 0, \pm1$). E-spin $SU(2)$ action equivalence of the 3 colors, by Chan–Paton reasoning, generates $SU(3)$ color symmetry. (In contrast to electrodoublet halves, conjugation of e-spin triplet halves yields nothing new.)

GUT Bosons

A final (12-member) boson family, that can be associated with the GUT extension of the Standard Model, is classifiable as $\{3,0|\ \cdot\ |2,1\}$ and $\{1,2|\ \cdot\ |0,3\}$. One half is an isodoublet and the other a color triplet. Electric charges are $\pm2/3$, $\pm1/3$, while $e = 0, \pm1$. Because the halves match those building the W triplet and the gluon octet, Chan–Paton vector couplings (boson events) exhibit $SU(5)$ symmetry. GUT bosons couple to lepton–quark pairs [10].

Quarks

A 72-member elementary-fermion family has wave functions classifiable as $\{3,0|\ \cdot\ |2,3\}$, $\{3,2|\ \cdot\ |0,3\}$, $\{3^*,2^*|\ \cdot\ |0^*,3^*\}$, $\{3^*,0^*|\ \cdot\ |2^*,3^*\}$. An unpinched half or antihalf with electro (pair) label 0 or 0^* carries color and

allows Yang–Mills $SU(3)$-symmetric coupling to gluons. As shown in Table 18.3, inversion (i.e., CP) connects L antiquarks with L quarks and R antiquarks with R quarks. (Quarks have electric charge $+2/3, -1/3$ while antiquarks have $Q = -2/3, +1/3$.) The chirality assignments of this table reflect the W–quark events of [10]. The action of these events, still under study, may deviate from the Standard Model because both pinched and unpinched quark halves are involved.

Table 18.3. Elementary quark wave functions. Halves with isospin are pinched. Each quark comes with one of 3 colors carried by its unpinched half

Q	Wave function	Elementary quarks
$+2/3$	$\{3^*, \Uparrow^* \mid \cdot \mid 0^*, 3^*\}$	$(\mathrm{u}, \mathrm{c}, \mathrm{t})_\mathrm{L}$
$-1/3$	$\{3, 0\mid \cdot \mid \Downarrow, 3\}$	$(\mathrm{d}, \mathrm{s}, \mathrm{b})_\mathrm{L}$
$-2/3$	$\{3^*, 0^*\mid \cdot \mid \Uparrow^*, 3^*\}$	$(\bar{\mathrm{u}}, \bar{\mathrm{c}}, \bar{\mathrm{t}})_\mathrm{L}$
$+1/3$	$\{3, \Downarrow \mid \cdot \mid 0, 3\}$	$(\bar{\mathrm{d}}, \bar{\mathrm{s}}, \bar{\mathrm{b}})_\mathrm{L}$
$+2/3$	$\{3, 0\mid \cdot \mid \Uparrow, 3\}$	$(\mathrm{u}, \mathrm{c}, \mathrm{t})_\mathrm{R}$
$-1/3$	$\{3^*, \Downarrow^* \mid \cdot \mid 0^*, 3^*\}$	$(\mathrm{d}, \mathrm{s}, \mathrm{b})_\mathrm{R}$
$-2/3$	$\{3, \Uparrow \mid \cdot \mid 0, 3\}$	$(\bar{\mathrm{u}}, \bar{\mathrm{c}}, \bar{\mathrm{t}})_\mathrm{R}$
$+1/3$	$\{3^*, 0^*\mid \cdot \mid \Downarrow^*, 3^*\}$	$(\bar{\mathrm{d}}, \bar{\mathrm{s}}, \bar{\mathrm{b}})_\mathrm{R}$

A 'handed' quark family is an e-spin nonet, the 3-valued e-spin of the pinched half distinguishing 3 'elementary quark generations'. A separate paper, dealing with chiral-fermion primitive rest mass, will consider a 3×3 generation-mixing matrix. Although e is conserved, quark color and generation are not separately conserved in weak interactions of quarks. Mixing of elementary quark generations occurs.

Leptons

Finally we identify leptons and antileptons as the 24-member chiral family with wave functions $\{3, \Uparrow \mid \cdot \mid 2, 1\}$, $\{1, 2\mid \cdot \mid \Uparrow, 3\}$, $\{3^*, \Uparrow^* \mid \cdot \mid 2^*, 1^*\}$, $\{1^*, 2^*\mid \cdot \mid \Uparrow^*, 3^*\}$. The unpinched half (antihalf) of each unstarred state is accessible, independently of the pinched half, for Yang–Mills L coupling to W bosons. The pinched half (antihalf), like that of a quark, carries a 3-valued generation label. As shown in Table 18.2, Standard Model elementary leptons and antileptons (with both L and R neutrinos and antineutrinos) match EQT basis quartet states. An Appendix B path constraint excludes a $\Downarrow$ or $\Downarrow_*$ pinched lepton half. (The pinched half is an isosinglet.)

The L and R families of Table 18.2 break as shown into isodoublet e-spin triplets with $Q = 0, 0, \pm 1$ and $e = 0, \pm 1$. Lepton (pinched-half) generation

associates with the three conserved e-spin values $e = 0, \pm 1$. There may nevertheless be mixing of generation $+1$ neutrinos (antineutrinos) with generation -1 antineutrinos (neutrinos), as well as mixing of 0-generation neutrinos and antineutrinos. Despite each of the (negatively charged) leptons e, μ, τ carrying a distinct and different value of conserved e, assignment of e to observed charged leptons will not be attempted here.

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19 Quantum Phenomena of Biological Systems as Documented by Biophotonics

Fritz-Albert Popp

In recent years more and more nations have started to invest millions of dollars in a new scientific field called biophotonics. From 2004 on, 50 million euros will be available in Germany alone for the development of this field, where the properties of photons from living tissues are the subject of investigation. Despite the fact that biological systems are known to react sensitively to single quanta – the eye, for instance, in its capacity for registering single photons with the highest possible resolution, is the most sensitive detector system for photons in the visible range – for a long time the effects of weak radiation from or onto living systems have been completely disregarded or underestimated. However, with the increasing sensitivity of the detector systems and increasing insight into the sensitive interactions of living systems and electromagnetic fields, it was gradually understood that nonthermal effects are likely to play a decisive role in understanding life. In particular, the well-known suggestions of Fröhlich, Prigogine, and others, the development of chaos theories, and even serious indications of macroscopic quantum coherence, opened the door to the new field of biophotonics, which includes not only new technical methods for understanding life by light, but also goes back to the roots of quantum theory in attempts to develop models of biological regulation from the cellular level up to the formation of consciousness.

This paper is a review of the historical, biological, but mainly also the physical roots of this branch of quantum biology, as demonstrated through the example of the existence and the meaning of single photons in living systems, not to be confused with ordinary heat radiation, but today called biophotons.

19.1 History of Biophotonics

As an outstanding developmental biologist of the third decade of the 20th century, the Russian scientist Alexander Gurwitsch [1, 2] tried to solve one of the most crucial problems of biology, i.e., the 'Gestaltbildungs' problem, which is the question of how living tissues transform and transfer information about the size and shape of different organs. Chemical reactions do not contain spatial or temporal patterns a priori. That was the reason why Gurwitsch looked for a 'morphogenetic field' which could regulate cell growth

and differentiation. In particular, in his so-called 'Grundversuch' (basic experiment), he found ample indication for the involvement of photons in the stimulation of cell division. Figure 19.1 shows this famous 'Grundversuch' of A. Gurwitsch.

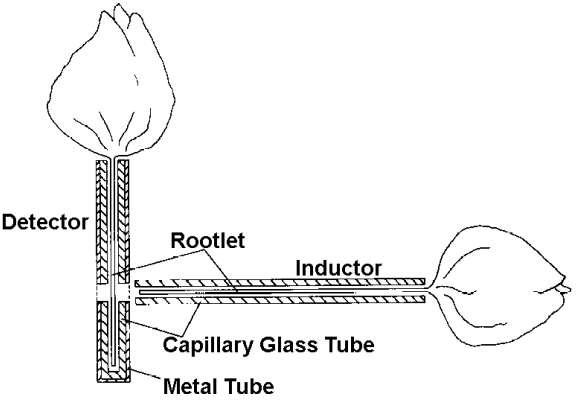


Fig. 19.1. Arrangement of Gurwitsch's experiment with onion roots

He used the stem of an onion root as a 'detector' and the tip of another one, very near to the detector but not actually touching it, as an 'inductor'. The subject of observation was the cell division rate at just the region of the stem where the tip pointed onto it. It turned out that the cell growth in this region of the stem did not change in the case of normal window glass being squeezed between the inductor and detector. However, as soon as the window glass was substituted with a quartz glass plate (which is transparent for UV light of about 260 nm), the cell division rate (number of mitoses) increased significantly. Gurwitsch interpreted this effect as the mitotic activity of single photons of about 260 nm triggering cell divisions. He called this photon emission from biological systems mitogenetic radiation, and repeated the experiments successfully with other biological systems, e.g., yeast.

However, despite confirmation of his results, also shown in a paper by the later Nobel laureate D. Gabor [3], the scientific community forgot Gurwitsch's work in view of:

- some (inessential) objections that came up,
- the rather difficult experimental work in this field involving a lack of appropriate photon-counting systems,
- a fast-developing biochemistry which tried to explain cell growth in terms of hormones and similar biomolecules.

Mitogenetic radiation was therefore considered as a kind of artifact.

After World War II, technical devices for measuring single photons improved through the development of photomultipliers. Russian biophysicists,

and others too, confirmed the existence of a 'dark luminescence' of all living systems in the visible range, which could not be explained in terms of heat radiation. The more viable work of the Russian groups who published mainly in the Russian journal *Biophysics* (translated in the USA) has been reviewed by Ruth [4]. However, in about 1970, the Russians stopped their activities in this field and turned to more practical questions about photosynthesis. Apart from biochemists in Poland associated with the researcher Slawinski [5], the Russian work received almost no attention.

In the Western world 'low-level luminescence' of living systems never became a serious subject of fashionable science. With the exception of groups associated with Inaba (Japan) [6], Boveris (USA) [7], and Quickenden (Australia) [8], this phenomenon of single photons from active biological tissues was completely disregarded or even thrown into disrepute. In cases where this non-thermal photon emission has been accepted at all, common opinion reflected the statements of the Russian biophysicist Zhuravlev [9] and the American chemist Seliger [10], i.e., their hypothesis that 'weak bioluminescence' originates from 'imperfections' in metabolic activity. This means that photons should occasionally be emitted since the living system is in the situation of a permanently excited state which tends to fall back into thermal equilibrium. Under these conditions, some scientists considered it obvious that highly reactive compounds such as radicals and oxidation reactants are the most likely candidates for photon sources.

19.2 Mitotic Figures

There are other biological phenomena that could have led to the realization that photons exist in living cells [11–18]. One is the well-known fact that about 10^5 chemical reactions per cell/per second take place. Without electronic excitation of at least one of the reaction partners, this would be impossible, and the number of thermal photons in the tiny reaction volume of a cell could never suffice to explain this high reaction rate. At least a factor of 10^{14} higher photon density in the optical range is necessary to provide this huge amount of chemical reactivity. Another point is Erwin Schrödinger's famous question concerning the surprisingly small number of aberrations in the migration of biomolecules during cell division. Let us look, for example, at the mitotic figures of a cell in mitosis (Fig. 19.2, left).

The only plausible answer to this question is the presence of cavity resonator waves (Fig. 19.2, right), which also provide the necessary stability of the molecular arrangements as the guiding forces for their movement. We calculated roughly the character of some transverse magnetic and electric modes and their wavelengths under the particular boundary conditions and for the dimensions of a cell, which may work as a conducting or dielectric resonant cavity (or both). Table 19.1 displays the list of results [19], where

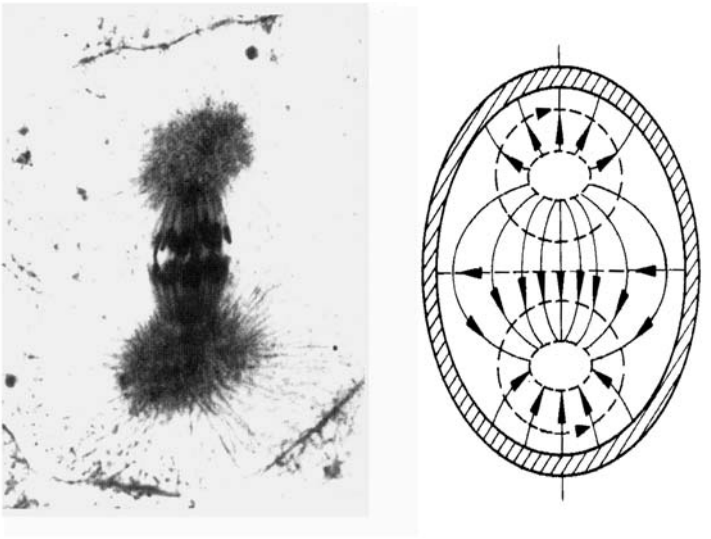


Fig. 19.2. *Left*: Completely developed spindle apparatus of a fish (Corregonus) in mitosis [38]. *Right*: Electric field of TM_{11} cavity modes in a right circular cylindrical cavity. Comparison with Fig. 19.8 (left) shows that mitotic figures are striking examples of long-lasting photon storage and coherent fields within biological systems [39]

the eigenvalues of the Bessel functions m, n correspond to the radial axis and p to the length of a right-circular cylindrical cavity.

The resonance wavelengths are in the optical range between 300 and 700 nm. We show in Table 19.1 that the dynamical structures of the mitotic figures during cell division can be obtained by superposition of cavity resonator waves of this kind. It indicates that the electromagnetic forces of these patterns present the most likely answer to Schrödinger‘s question of why the error rate vanishes.

It is evident that there is no workable way to measure these quasi-standing light waves directly within the intracellular space. However, if one puts a sufficiently highly sensitive photomultiplier in front of the living tissue, then one expects to measure at least single photons in the visible range which should display spatial and temporal correlations with biological functions, i.e., cell growth.

In agreement with these considerations, around 1970, my interdisciplinary group of physicists, biologists and physicians at the University in Marburg (Germany) found significant correlations between some optical properties of biomolecules (including polycyclic hydrocarbons) and their biological efficacy (including carcinogenic activity) [4, 11–14]. The basic question came up as to whether the excited states of biomolecules could be responsible for the light emission in biological tissues or whether a photon field in living systems is the

Table 19.1. Modes of a cylindrical cavity of the same dimensions as typical cells

TE mode mnp	TM mode mnp	Wavelength λ/mn	Number of stored photons $\frac{E_0^2 \varepsilon V_{\text{res}}}{8\pi} \times 10^{10}$erg
111		690	4.900
	010	574	7.787
112		571	5.929
	011	546	4.099
	012	481	4.655
113		462	7.323
211		438	1.589
	013	410	5.451
212		402	1.730
114		379	8.910
	110	360	2.939
213		358	1.943
011	111	353	3.004
	014	349	6.407
012	112	333.5	3.181
311		323	0.778
115		318	10.606

regulator for the excitation of the biological matter. This problem is similar to the question: Which came first? The chicken or the egg?

In contrast to the purely biochemical point of view, this search for the original regulator could be approached in terms of information transfer in biological systems supported by an increasing understanding of quantum optics, in particular in the non-classical range. First, I will confine myself to the most essential experimental results that have been obtained from this time on by careful investigation of 'low-level luminescence' or, as we have called this phenomenon, biophotons. Then, I will show that the more physical basis of interpretation provides a rather consistent picture of this universal phenomenon of weak photon emission from living systems. Lastly, some theoretical implications will be discussed.

19.3 Measurements of Essential Properties of Biophotons

Biophotons are measured by detectors based on photomultiplier techniques. These instruments provide both high sensitivity and high resolution. Our single photon counting system functions at a sensitivity of about 10^{-17} W and a signal-to-noise ratio of at least 10. The cathode of an EMI 9558 QA photomultiplier is sensitive within the range of 200 to 800 nm. The noise

is reduced by inserting the multiplier into a cooling jacket, where copper wool provides thermal contact. In addition, a grounding metal cylinder protects the multiplier from electric and magnetic fields. In order to prevent the multiplier from freezing, the whole tube together with the cooling jacket is kept in a vacuum. Thus, the quartz glass in front of the multiplier tube has no thermal contact with the cooled cathode and cannot become covered with moisture. An optimal cooling temperature is produced at about $-30°\mathrm{C}$. With the use of a chopper, the equipment is able to register a real current density of 2 photons/s cm^2 at a significance level of 99.9% within 6 hr. A detailed description of the method has been presented elsewhere [4]. Figure 19.3 displays an implementation of the equipment.

We only report here results that have been reproduced several times and confirmed by different groups. Thus the essential characteristics of biophoton emission may be summarized as follows [15–17]:

- The total intensity i from a few up to several hundred photons/s cm^2 indicates that the phenomenon is quantum physical, since fewer than about 100 photons are ever present in the photon field under investigation.
- The spectral intensity $i(\nu)$ never displays small peaks around definite frequencies ν. Rather, the quite flat spectral distribution has to be assigned to a non-equilibrium system whose excitation temperature $\theta(\nu)$ increases linearly with the frequency ν. This means that the occupation probability $f(\nu)$ of the responsible excited states does not follow a Boltzmann distribution $f(\nu) = \exp(-h\nu/kT)$, but the rule $f(\nu) = \text{constant}$ (Fig. 19.4).
- The probability $p(n, \Delta t)$ of registering n biophotons ($n = 0, 1, 2, \dots$) in a preset time interval Δt follows, under ergodic conditions, a Poissonian distribution surprisingly accurately: $\exp(-\langle n\rangle)\langle n\rangle^n/n!$, where $\langle n\rangle$ is the mean value of n over Δt. This holds true at least for time intervals Δt down to 10^{-5} s. For shorter time intervals Δt, no results are currently available (see Fig. 19.5) [18].
- After excitation by monochromatic or white light, the 'delayed luminescence' of every biological system relaxes quite slowly and continuously down to 'spontaneous' biophoton emission, not according to an exponential function, but with a strong relationship to a hyperbolic-like $(1/t)$ function, where t is the time after excitation (Fig. 19.6).
- The optical extinction coefficient of biophotons passing through thin layers of sea sand and soya cells of various thickness can have values of at least one order of magnitude lower than that of artificial light with comparable intensity and spectral distribution, indicating that this difference cannot be explained in terms of wavelength dependence on extinction [20].
- The biophoton emission displays the typical temperature dependence of physiological functions, such as membrane permeability, glycolysis, and many others. This means that with increasing temperature one gets overshoot reactions, while with decreasing temperature an undershoot response may take place. The resulting 'temperature hysteresis loops' of

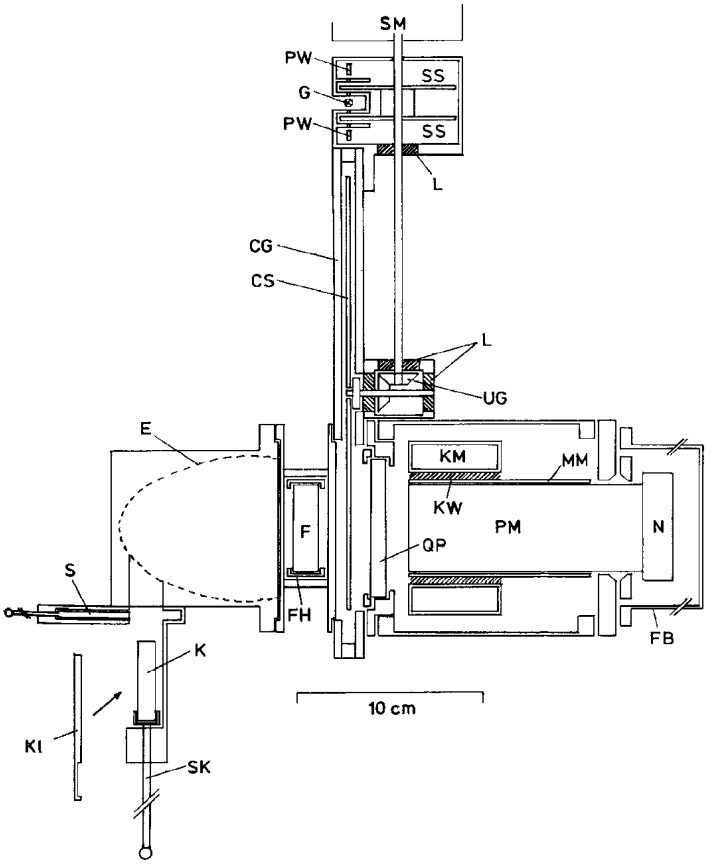


Fig. 19.3. Measuring equipment. PM photomultiplier, CS chopper disc, CG housing of chopper disc, E ellipsoid, F filters, FB Faraday cup, FH filter holder, G lamp, K test quartz glass, Kl flap, L ball bearing, KM cooling jacket, KW copper wool, MM metal cylinder, N network, PW photosensitive resistor, QP quartz glass, S slide to close ellipsoid, SK rod to move up test glass, SM, UG geared motor, SS sector discs

biophoton emission (Fig. 19.7) can be described by a Curie–Weiss law dependence [21].

- Reactions to stress are frequently indicated by an increase in biophoton emission.
- There is evidence that the conformational states of DNA influence biophoton emission. This has been demonstrated, for instance, by the intercalation of ethidium bromide (EB) into DNA (Fig. 19.8) According to the winding up and renewed unwinding of DNA by increasing concentrations of EB, the biophoton emission intensifies and drops down in a rather

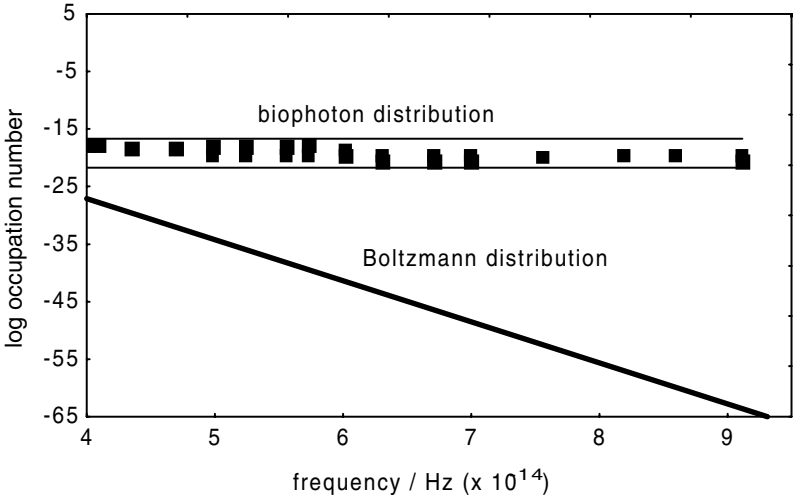


Fig. 19.4. In the case of average occupation numbers, we obtain an f = const. distribution which displays increasing deviation from the Boltzmann distribution with increasing frequency

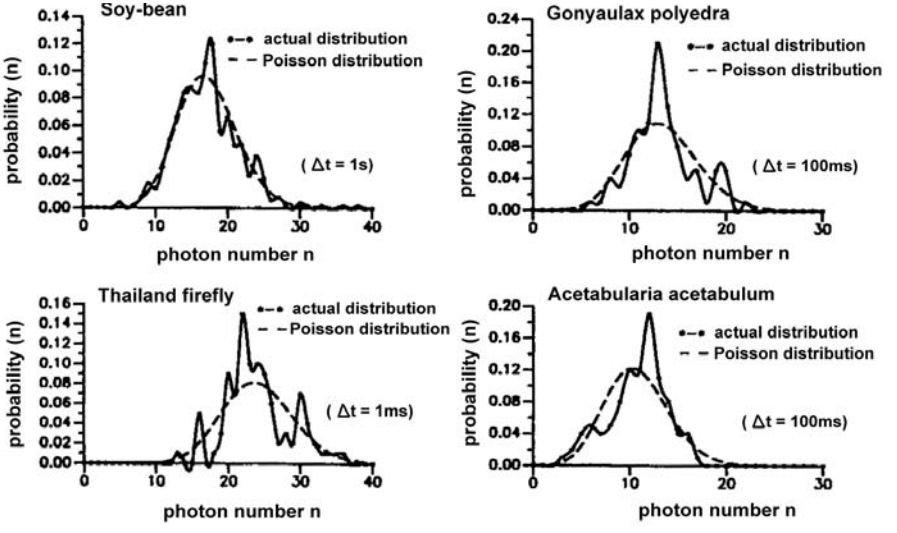


Fig. 19.5. The photocount statistics [= probability $p(n, \Delta t)$ of registering n counts in a preset time interval Δt, where $n = 0, 1, 2, \ldots$] is very similar for all biological systems. If Δt is so small that the mean number of photons in the field becomes lower than about 100, $p(n, \Delta t)$ displays a Poissonian (and sometimes even a sub-Poissonian) photocount distribution. There are 4 different examples with different Δt. 100 measurement values have been used for evaluation in each case, where the biological state was kept quasi-stationary

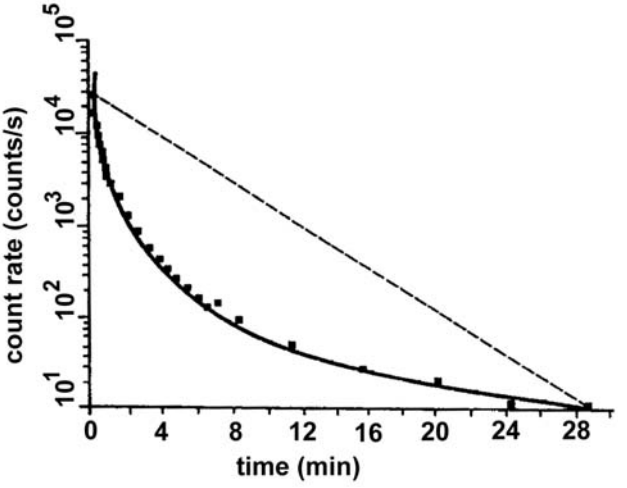


Fig. 19.6. Instead of an exponential decay (*dashed line*), living cell populations (here tissue of Bryophyllum daigremontanum) exhibit a hyperbolic relaxation of photon intensity after exposure to white-light illumination. This holds for total as well as for spectral observation (here at 676 ± 10 nm). Under ergodic conditions, hyperbolic decay is a sufficient condition for perfect coherence

strong correlation. This and other results indicate that chromatine is one of the most important sources of biophoton emission [22, 23].

The Poissonian distribution of photocount statistics $p(n, \Delta t)$ under ergodic conditions together with the hyperbolic relaxation function of delayed luminescence is a sufficient condition for a fully coherent photon field [24].

Thus we can conclude that biophotons originate from a coherent field. Before we discuss the theoretical aspects, let us look at some biological phenomena for which there is a rather plausible explanation but which cannot be understood in terms of common molecular biology.

19.4 Biological Impacts

Once the coherence of biophotons is accepted, it is not difficult to predict a variety of biological phenomena which deviate considerably from the 'conventional' point of view, thus providing a reliable basis for examining the theory and for obtaining a more profound understanding of biology.

It is evident that coherent fields give rise to destructive and constructive interference, in which, considering the energy conservation law, zones of destructive interference have to be compensated for by zones of constructive interference (Fig. 19.9). According to the theory due to R. Dicke [25], there is a preference for constructive interference ('super-radiance') in the initial

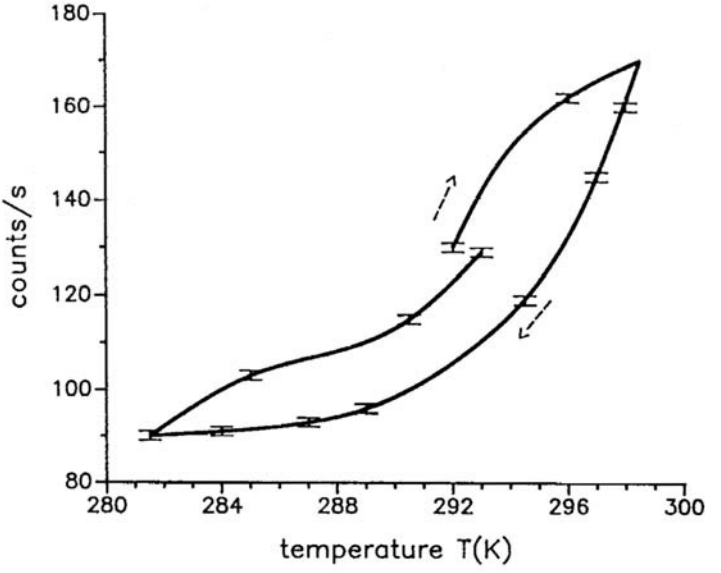


Fig. 19.7. The biophoton intensity of living tissues shows a hysteresis-like dependence on the temperature T, if T is cyclically varied. The example shows etiolated barley, 4 days of germination. The variation of temperature starts at $T = 292$ K with the rate $\partial T/\partial t = 0.5$ K/min. At $T = 298.5$ K the rate of temperature change is reversed to $\partial T/\partial t = -0.5$ K/min, and again at $T = 281.5$ K with $\partial T/\partial t = 0.5$ K/min. This hysteresis-like behaviour of biophoton intensity can be accurately described as a Curie–Weiss law dependence

phase of the interaction between radiation and non-randomly oriented matter of suitable size, while destructive interference ('sub-radiance') dominates after longer periods of time. Consequently, there is always a considerable probability of destructive interference in the biophoton emission of living systems in the space between the living cells.

This means that the biophoton intensity of living matter cannot increase linearly with the number of units, but has to follow the effective amplitudes of the interference patterns of the biophoton field between living systems. A striking example is the measurements on daphnia [26, 27].

In darkness, daphnia magna Strauss were put into water at 18°C within the quartz cuvette of the biophoton measuring equipment. We altered the numbers n of daphnia from 1 to 250, always selecting animals of about equal size. After each alteration, the intensity of the biophoton emission was registered. Since every one of the inbred animals emits almost the same intensity, one expects a dependence of biophoton intensity on the number of animals, like that displayed in Fig. 19.10a. After correction for self-absorption, it should not significantly deviate from that of Fig. 19.10a. However, careful measurements showed evidence of the results displayed in Fig. 19.10b.

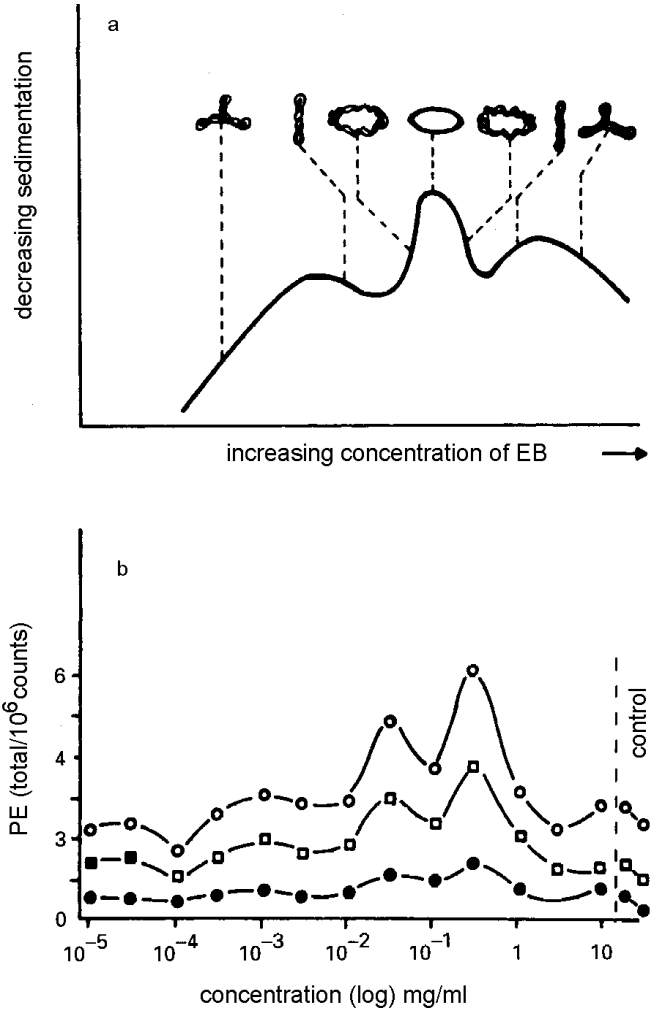


Fig. 19.8. (**a**) The more ethidium bromide (EB) is added, the more EB molecules are inserted between the base pairs of DNA. This intercalation leads to the unfolding of the helix structures of the DNA. The degree of this unfolding is experimentally determined by the sedimentation of the DNA. After complete unfolding, further insertion of EB leads to a new upwinding of the DNA helix in the opposite direction. (**b**) The observation of the biophoton emission after adding EB shows (*lower curve*: after 1 hr) that the intensity displays the same dependence on the concentration as in (**a**). This typical profile becomes even more evident after longer measurement time (*middle curve*: after 3 hr, *upper curve*: after 5 hr), which indicates a dependence of the biophoton emission on the spatial structure of DNA

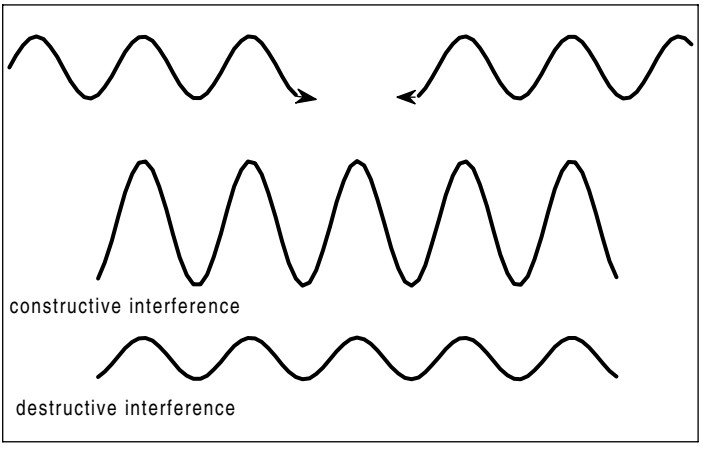


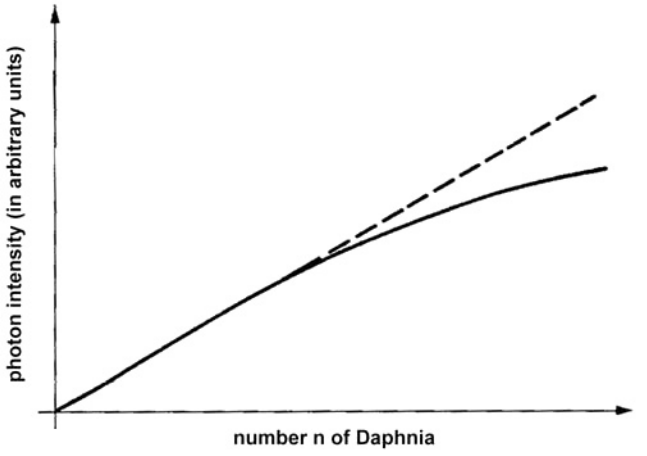
Fig. 19.9. If two waves interfere, the phase relations will in general lead to zones where they amplify mutually (constructive interference) or alternate (destructive interference). For coherent fields, these processes provide a basis for regulation and communication

The results from interference patterns of biophotons between the animals under investigation were as expected. There is a tendency for destructive interference resulting in a lower intensity than expected from a linear increase. The most efficient destruction of the biophoton field outside the animals is obtained with about 110 animals, corresponding to the population density of daphnia in free nature. This zone of most efficient destruction according to the energy conservation law is at the same time the zone of highest efficacy in 'storing' light within the animals.

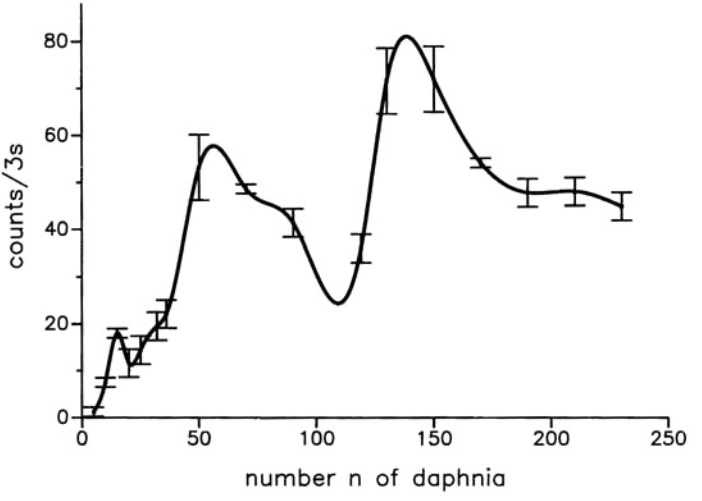
To some extent one is justified in saying that living systems 'suck' the light away in order to establish the most sensitive platform of communication. A more detailed description of this phenomenon has been presented elsewhere [28]. Actually, this biocommunication by means of mutual interference of the biophoton field provides necessary information about the equality or difference of species, since similar animals have similar wave patterns. The signal-to-noise ratio becomes optimized as soon as the wave patterns interfere under maximum destruction between the communicating systems, since every perturbation then leads to an increase (signal) that the connected systems have to become aware of.

This rather ingenious means of biocommunication provides the basis for orientation, swarming, formation, growth, differentiation, and 'Gestaltbildung' in every biological system [17].

On the other hand, as soon as this capacity for coherent superposition of modes of the biophoton field (where longer wavelengths may also be included) breaks down, in the first stage of destruction, one expects a consequent increase in biophoton emission (or delayed luminescence) with increasing num-



(a)



(b)

Fig. 19.10. (**a**) If photon emission from daphnia is dependent on mutual interactions of the animals, one expects a linear increase with increasing number of daphnia. This linearity will show a small decline for a large number of animals as soon as self-absorbance has to be taken into account. (**b**) Mean values of the photon intensity of adolescent daphnia in 15 ml volume with the weighted standard deviation

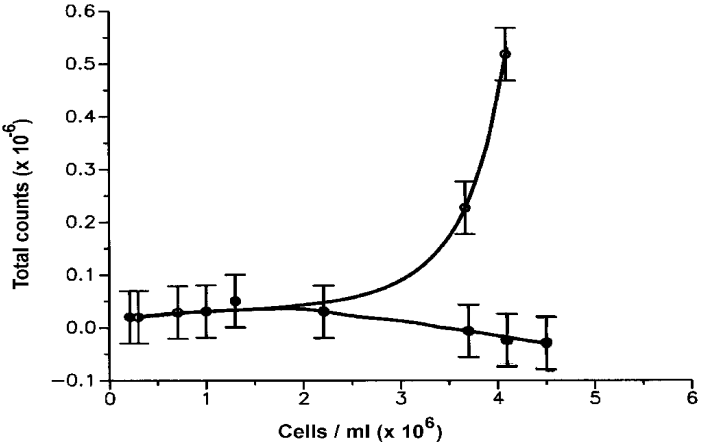


Fig. 19.11. 'Delayed luminescence' from tumor cells (*upper curve*) and normal cells (*lower curve*), as measured by Schamhart. The different curves can be approximated by a non-linear (cubic) dependence of intensity on cell number n

bers of living units within a biological population. This was first confirmed by D. Schamhart [29] (Fig. 19.11) and Scholz et al. [30] (Fig. 19.12). Actually, tumor cells lose the capacity for destructive interference according to their loss of coherence. At the same time, delayed luminescence moves from the hyperbolic-like relaxation of normal cells to the exponential one of tumor cells.

A further striking example is the synchronous flickering of dinoflagellates (Fig. 19.13). As soon as these animals see each other, their bioluminescent flickering decreases and displays significantly more synchronous light pulses than in the case when they are separated from each other [31]. This phenomenon can be explained in terms of chemically amplified biophoton emission (which is called bioluminescence), establishing destructive interference as soon as the animals 'see each other' and displaying synchronous pulses as a consequence of the disruption of the destructive interference patterns.

Even bacteria seem to use this kind of 'communication' within their nutrition media [32]. Figure 19.14 displays one of the measurements on Enterococcus faecalis. Growing bacteria emit such low biophoton intensity that it cannot be registered, in contrast to the permanent photon emission of their nutrition media. (It is impossible, by the way, to produce nutrition media without spontaneous photon emission, originating from oxygenation processes.) At a definite number of bacteria, the total intensity of the system drops as a consequence of active photon absorption by the bacteria within the medium. At higher numbers of bacteria, it may then happen that this absorbance disappears.

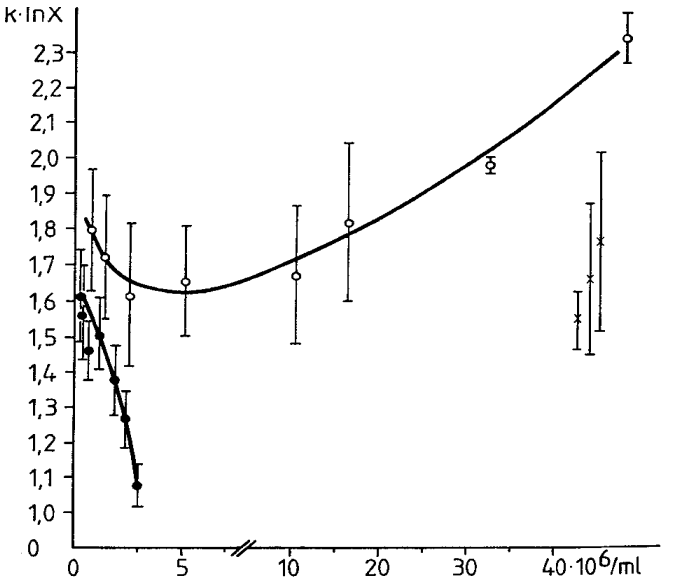


Fig. 19.12. The decay parameter of the hyperbolic approximation that is adjusted to the relaxation dynamics of the afterglow of different cell suspensions after exposure to weak white light illumination is shown versus cell density. The *lower curve* displays the improvement of hyperbolic relaxation of normal amnion cells with increasing cell density. The *upper curve* shows the opposite dependence exhibited by malignant Wish cells. The three measurements *on the right* of the figure correspond to the nutritive medium alone

Again, the destructive interference of bacteria within the coherence volume of the light-emitting nutrient molecules provides an explanation for this obviously rather universal process in living nature.

It should be noted that growth regulation through biophoton emission has to follow a law where in addition to linear stimulation ($\dot{n} \propto n$) a nonlinear inhibition ($\dot{n} \propto n^2$) has to take place. Consequently, the correlation between growth rate and biophoton emission should be based on such a relationship. Figure 19.15, as a result of measurements, confirms this connection.

Recently, it has been shown experimentally that in accordance with the presumptions of Bajpai [33], and Gu and Li [17], living systems are even able to emit squeezed light [34]. This leads to new grounds for establishing the theoretical basis of biophoton emission.

19.5 Theoretical Approach

Let us start with a striking example of coherence which demonstrates the whole essence of this subject. The leaf crassula ovata was excited by a He–Ne

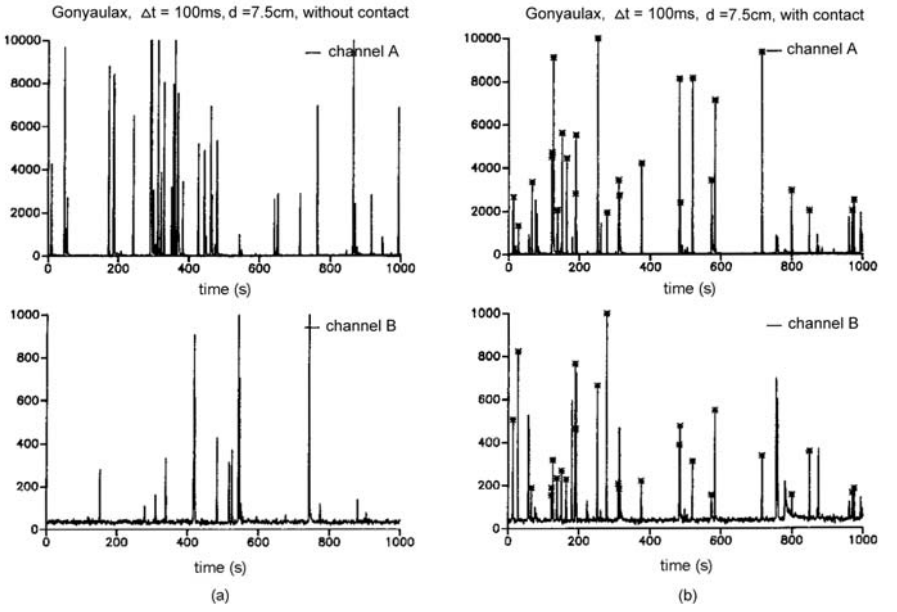


Fig. 19.13. If ones separates two cultures of dinoflagellates, their bioluminescence flickering is completely asynchronous (*left*). As soon as they are in optical contact, a large amount of flickering is synchronous (*right*). *Stars* indicate synchronous flashes

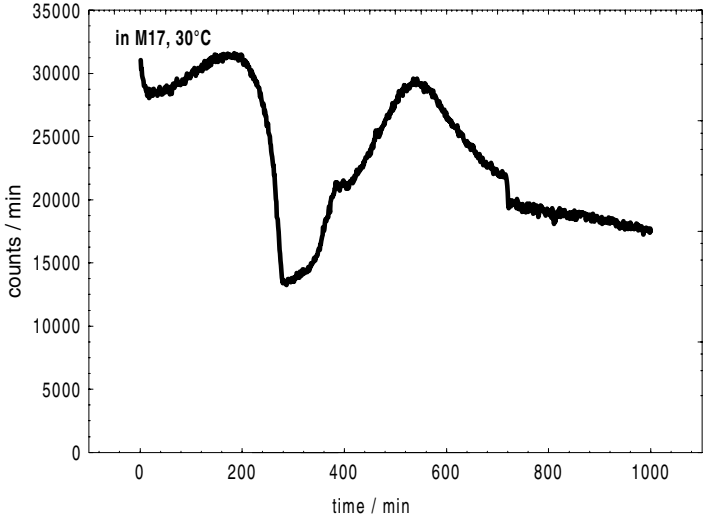


Fig. 19.14. Growing bacteria in a culture medium, which by oxidative reactions always emits light, absorb the light of the medium from a definite density. For higher densities this absorbance may decrease

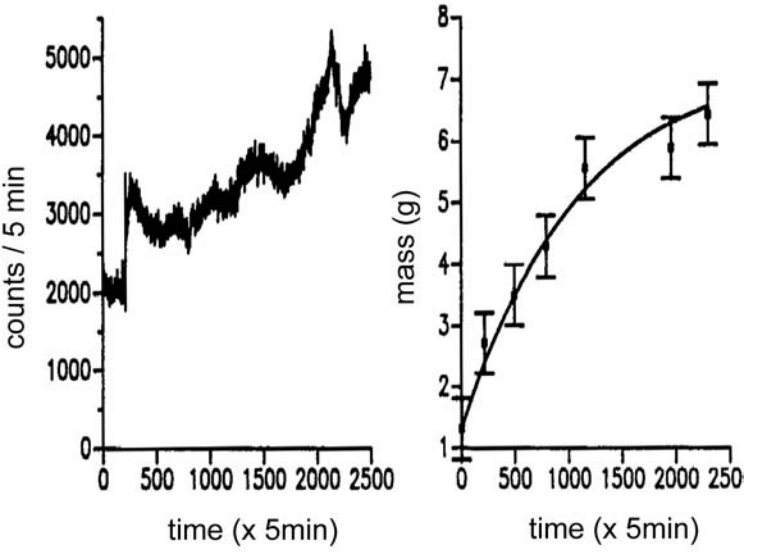


Fig. 19.15. The biophoton intensity from 9 soy seeds increases during germination in darkness, as shown on the *left*. The mass m during germination increases according to the curve on the *right*, where the measured values (*points* with *error bars*) have been approximated by the growth curve $m = A/[B + (1 - B)\exp(-Ct)]$, with $A = 1.3$ g, corresponding to the mass of 9 soy seeds, $B = 0.178$, $C = 2.6 \times 10^{-4}$ min^{-1}, and t representing time

laser (wavelength 632 nm) after 10^4 times attenuation by grey filters with an intensity of about 10^6 photons/s. The exposure time was varied between a few seconds up to 200 s. As a result, the delayed luminescence intensity increased and, from about 100 s exposure time onwards, distinct oscillations appear around the hyperbolic-like relaxation function $I(t) = a/(1+bt)$, where $I(t)$ is the biophoton intensity after excitation at time t, and a and b are constant values. Figure 19.16 shows a typical case of delayed luminescence and Fig. 19.17 displays the oscillation around the hyperbolic-like relaxation and a mathematical approximation in terms of the function

$$\Delta I(t) = \frac{c}{1+dt} \exp\left[\mathrm{i}\gamma \ln(1+\nu t) + \phi\right] ,$$

where $c = 30\,000$, $d = 0.5$, $\gamma = \pi/\ln 3$, $\nu = 0.5$ and $\phi = 0$.

This phenomenon of oscillations around the relaxation function of delayed luminescence was observed and described for the first time in 1980 on the leaf bryophyllum daigremontanum [15], and in 2002 it was shown in the case of the single cell Acetabularia acetabulum that it can be traced back to a coupling of coherent states of the biophoton field [35]. Recent investigations on the 'hyperbolic oscillations' of the leaf crassula ovata showed evidence that this phenomenon is not of molecular origin [36]. The light that corresponds

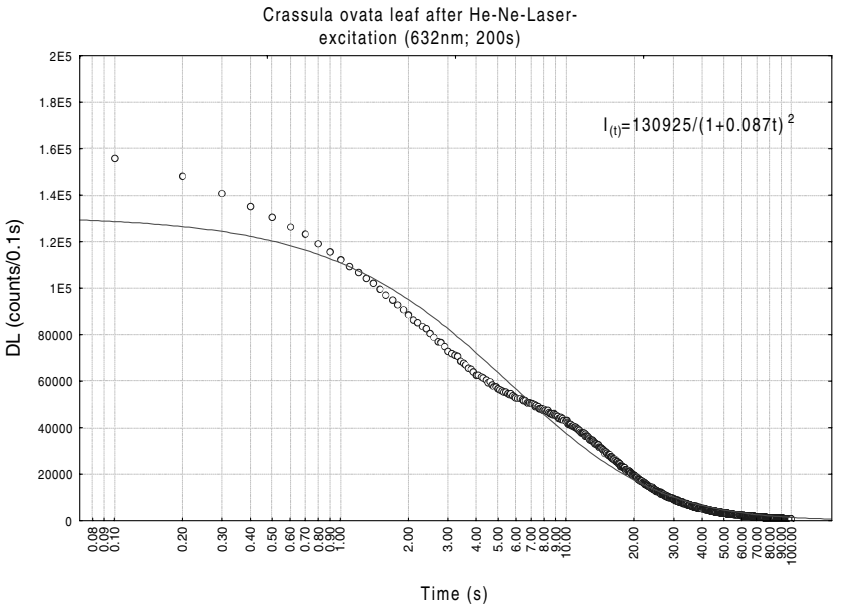


Fig. 19.16. Delayed Luminescence of a crassula ovata leaf, excited by a He–Ne laser for 200 s. The relaxation function is approximated by a hyperbolic-like decay function, satisfying the differential equation for coherent states. The residuum displays 'hyperbolic' oscillations around the continuous decrease

to this photon emission has a wavelength of about 750 nm. But with the oscillation, the light emission itself disappears completely as soon as the leaf is homogenized down to particles smaller than cellular size. Consequently, this phenomenon reflects coherent and collective excitations on the cellular and super-cellular levels, which might form the basis of intercellular communication by coherent states of biophotons.

A further more refined investigation of these oscillations shows that the general solution of the intensity of delayed luminescence is

$$I(t) = a\left[\prod_j (1+b_j t)^{P_j}\right] \sin\left\{\left[\sum_j \gamma_j \ln(1+\nu_j t)\right] + \phi\right\} .$$

The surprising fact about the phenomenon is not only the high degree of coherence and the non-molecular basis for the essence of life in terms of quantum theory, but rather the high stability of frequency-locking and phase-locking in biological systems. Figure 19.18 displays the ratio of the approximated mathematical function to the measured values and its approximation by a tangent function, which shows evidence of phase-locking of the oscillations over at least a few minutes. A measurement with this accuracy is not possible with artificial laser light or classical light.

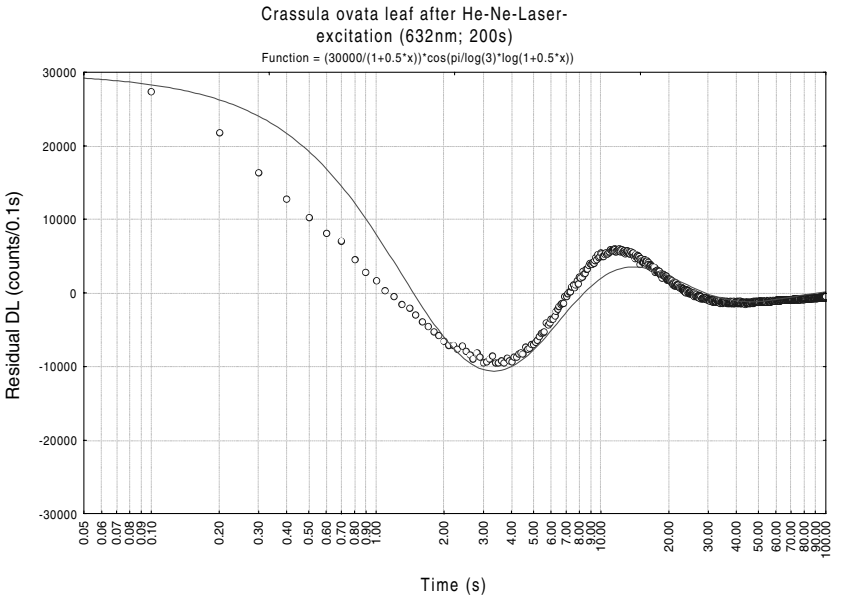


Fig. 19.17. The oscillating residuum function of Fig. 19.16 can be approximated by the function $c/(1+dt)\exp[\mathrm{i}\gamma\ln(1+\nu t)+\phi]$, where all the parameters with the exception of time t are constant

Consequently, the theory of biophoton emission refers not only to classical electrodynamics and thermodynamics but also to quantum theory. Experimental starting points for biophoton theory are:

1. the spectral intensity of biophoton emission and its temperature behavior [15, 21],
2. the photocount statistics [15],
3. the hyperbolic-like delayed luminescence relaxation [15],
4. hyperbolic oscillations around the relaxation curve [15, 35],
5. coupling of the different modes [17],
6. the squeezing into both branches of minimum uncertainty wavepackets, i.e., minimization of position and of momentum uncertainty [34],
7. the strong correlation with DNA dynamical states [22].

From a biological point of view, for instance, the mitotic figures [19], the 'interference structure' of biophoton emission from daphnia [26], the qualitatively different photon emission and reemission of tumor tissue and normal tissue [15], and the correlation with growth and differentiation of cells [37] will all become understandable under the umbrella of biophoton theory.

The mean value of the number n of photons of energy $h\nu$ of a homogeneous electromagnetic field with amplitude E_0 can be estimated by equating the energies $nh\nu$ of the photons and $\varepsilon_0|E_0|^2V/8\pi$ of the field, where ε_0 is the

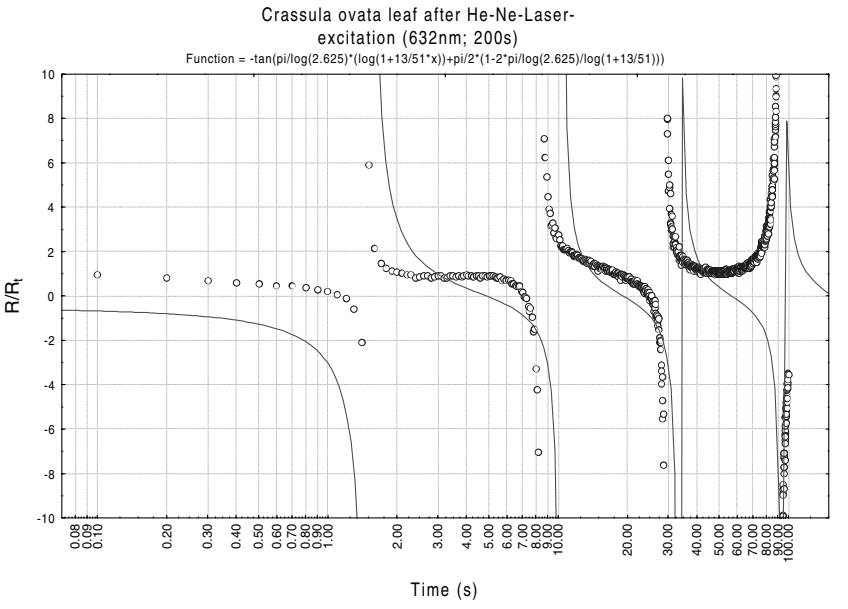


Fig. 19.18. The ratio of the approximated theoretical function of Fig. 19.17 to the experimental values displays tangent-like functions with surprisingly stable phase relations (see text). The continuous hyperbolic-like relaxation function plus the oscillatory function are solutions of the coherent-state differential equation under the boundary condition of energy conservation (homeostatic response)

dielectric constant and V the volume of the field. For one photon in the optical range of say 3 eV, one gets a field amplitude on the order of 10^6 V/cm over a volume of a cell of about 10^{-9} cm^3. This means that in a situation where the electric field amplitudes of the cavity modes stabilize the mitotic figures in the range of 10^6 V/cm (corresponding to about the membrane field components), only one photon in the optical range would suffice for this effect. In other words, the low intensity of biophoton emission may well reflect its biological functions in cells, such as stabilization of biomolecule migration, transportation of angular momentum for rotating DNA during replication or transcription, but also provision of the chemical reactivity of about 10^5 reactions per cell per second, always occurring at the right time and at the right place.

The resonator model is one of the most powerful approaches for understanding biophoton emission. In fact, living systems may be looked upon as the most stable form of matter through the storage of sunrays. To optimize what we call life, the gradient between the high temperature of the sun and the low one of the earth could be a necessary condition for life, particularly the prolongation of the entropy increase of light into heat, which means optimization of the storage capacity for sunlight. Photosynthesis, the process

providing the elementary food supply of plants, is a striking example. Let us remember that here also there is a clear connection between the resonator value of a cavity and its information content, pointing to a key understanding of biological systems in terms of informational rather than energetic 'engines'. Furthermore, the resonators may develop nonlinear capacities just because of their low photon emission. The deviation from the classical Q-value of the typical resonator may then take the form

$$Q^* = \frac{Q}{1-C} \,, \tag{19.1}$$

where Q^* is the resonator value of the quantum coherent resonator, Q is the value of the classical 'chaotic' resonator, and C describes the ratio of the quantum coherent energy distribution of the resonator to the totally available (chaotic + coherent) energy. This kind of resonator may develop rather high storage time ($Q^*/Q \to \infty$ for $C \to 1$), but may be able to emit or to remove photons actively for $C > 1$. It also describes Bose-condensation-type phenomena, as Fröhlich has postulated. This can be seen in the following way. Take the Bose–Einstein distribution of the spectral photon density (number of photons per unit volume and wavelength λ) at temperature T:

$$N(\lambda) = \frac{8\pi}{\lambda^4} \frac{1}{\exp \dfrac{\varepsilon - \mu}{kT} - 1} \,, \tag{19.2}$$

where $\varepsilon = hc/\lambda$ is the photon energy, μ the chemical potential, and k Boltzmann's constant. The chemical potential is defined as $\mu = -T(\partial S/\partial n)_{E,V}$, where $\mathrm{d}S$ is the entropy change through absorption of a photon. Figure 19.2 tells us that the absorption of a biophoton by the multiplier outside the system ($\mathrm{d}n < 0$) leads to an increase in the entropy of the system, and consequently to a value $\mu > 0$. In the case where there is no entropy loss by thermal noise, we then have $\mu = \varepsilon$. In the real case we may write

$$\mu = \varepsilon - kT \ln W \,, \tag{19.3}$$

where W corresponds to the thermodynamical probability of the photons under investigation. Inserting this in (19.2) results in

$$N(\lambda) = \frac{8\pi}{\lambda^4} \frac{1}{W-1} \,. \tag{19.4}$$

Now we see clearly the Bose condensation effect of a Fröhlich mode according to $W \to 1$ as well as the connection with the corresponding value C in (19.1). $C = 1$ implies that the whole energy of the system, with the exception of that of classical currents, belongs to a coherent field. In that case we get a resonance-like absorption of photons in the mode $W \to 1$. If we include the possibility of 'squeezed' light, we may even describe removal of photons by $W < 1$ or the extension of W, where the thermodynamical probability of

the photon field corresponds to the vanishing chemical potential according to (19.3):

$$\ln W = \frac{\varepsilon}{kT} \,. \tag{19.5}$$

In this case we once again have the spectral intensity of thermal radiation.

However, the average spectral intensity of biophoton emission is a further indicator of its real nature. W turns out to be rather constant and independent of the wavelength (see Fig. 19.4). For all biological systems, one finds the order of magnitude of W in the band between 10^{17} and 10^{23}, which is certainly far from thermal equilibrium. This constancy of W [or $f = 1/(W-1)$] invites us to postulate that:

- living systems keep this rule $W =$ constant over the whole spectral range up to a limiting frequency ν_0, corresponding to a cutoff wavelength $\lambda_0 = c/\nu_0$, where c is the velocity of light,
- W is adjusted in living systems in such a way that in a biological equilibrium state – which is far from thermal equilibrium – the whole of the available thermal spectral energy is equally distributed over all the available resonance modes of the biological system.

These statements (Fig. 19.4) show that the biological system is some kind of information engine that transforms heat energy into the occupation of coherent modes by use of food supply, i.e., sunrays. It optimizes its energy content by adjusting it to the thermal boundary conditions of a heat bath, probably by isoenthalpic processes. At the same time, this balance between the thermal energy density and the nonthermal occupation of the modes explains the continuity of biological evolution from equilibrium systems to open ones.

Consequently, we enunciate

$$\frac{1}{\langle W\rangle}\int \frac{8\pi}{\lambda^4}\frac{hc}{\lambda}\,\mathrm{d}\lambda = \frac{2\pi}{\langle W\rangle}\frac{hc}{\lambda_0^4} = \int \frac{8\pi}{\lambda^4}\frac{hc}{\lambda}\frac{1}{\exp\dfrac{hc}{kT\lambda}-1}\,\mathrm{d}\lambda = \frac{8}{15\pi^5}\frac{(kT)^4}{(hc)^3}\,, \tag{19.6}$$

where $\langle W\rangle$ is the average of W over all the modes of the biological resonator system, and the integration on the left-hand side runs from ∞ to λ_0, but on the right-hand side runs from ∞ to 0. Equation (19.6) provides the relation between $\langle W\rangle$ and the cutoff wavelength λ_0, that is:

$$W = \frac{15}{4\pi^4}\left(\frac{hc}{kT\lambda_0}\right)^4 \,. \tag{19.7}$$

We know that the spectral biophoton intensity is on the order of a few up to several hundred photons per cm^2, and it is in the range from 200 to 800 nm, corresponding to a $\langle W\rangle$ value between, say, 10^{17} and 10^{23} (see Fig. 19.4).

Insertion into (19.7) teaches us that the corresponding λ_0 is on the order of a few angstrom units. It fits into our images of the smallest size of a resonating structure within a biological system, because the smallest possible resonators are of this order of magnitude, i.e., the distances between neighboring base pairs of the DNA. At the same time it once again supports the exciplex model of biological evolution, which has already been discussed on several occasions.

A corresponding model concerns the adiabatic or isoenthalpic expansion of a photon gas, initiated by sunrays in the smallest possible resonance cavities of a biological system and expanding more and more to the bigger-sized ones by the f (or W) = const. rule, down to a final thermal degradation in the ULF ranges, where, with increasing evolutionary states, the number of resonator modes increases by shifting down also toward lower and lower boundary frequencies. According to the Louisell noise theory, the extension of resonating frequencies increasingly protects against thermal damping of a coherent system.

A further important point about the thermodynamics of biophoton emission is that the entropy S of the open system with $\mu = \varepsilon - kT \ln W$ becomes independent of temperature T. In fact S may even increase to values that are higher than that of thermal equilibrium systems, because the number of modes increases as $1/\lambda_0^3$. A straightforward calculation shows that the entropy is higher than the equiblibrium state as soon as

$$W > \frac{15}{4\pi^4} \left(\frac{hc}{kT\lambda_0} \right)^3 . \tag{19.8}$$

Comparison with (19.7) shows that this case is generally fulfilled as soon as the system relaxes to its steady state. However, as soon as the modes are coupled, the number of modes may decrease in such a way that the entropy becomes much lower than that of the equilibrium state. Theoretically, it may even reach the value 0. This important property of the system, to vary between a state of higher entropy than the thermal equilibrium state and one of lower entropy, explains both the stability and sensitivity of biological systems that has been discussed elsewhere. At the same time, this result provides a fundamental explanation of what we call homeostasis.

While the spatial pattern of the electromagnetic resonance modes is determined by Maxwell's equations, the dynamics is subject to quantum theory. One should note that, even if the light in cells originated from a chaotic field, the volume of a cell is always within the coherence volume of chaotic light. The coherence length of chaotic light from electronic transitions of molecules is the lifetime τ times the velocity of light and, even for allowed optical transitions, is much longer than the typical dimensions of a cell. This means that it is practically impossible for photons to lose their phase information over the distance of a biological cell. Consequently, it is impossible to determine the molecular source of biophotons, because even in the case of chaotic states (which may certainly contribute to the overall emission), the whole cell is

subject to the coherence volume, and the localization of the origin of a photon is not possible within this range. In other words, biophotons are in any case characterized by their relatively high degree of coherence within the volume of their activities. However, there are more than mere indications that the origin of biophotons is a fully coherent field, following the equation

$$a|\alpha\rangle = \alpha|\alpha\rangle \ , \tag{19.9}$$

where a is the annihilation operator, and $|\alpha\rangle$ and α are the coherent state and its eigenvalue (field amplitude), respectively.

That biological systems are governed by quantum coherent states has been shown by:

- the Poissonian photocount statistics of biophoton emission, which is a necessary condition for a fully coherent field,
- the hyperbolic-like relaxation of delayed luminescence, which is a sufficient condition for a fully coherent field under ergodic conditions.

The ergodic condition, on the other hand, has been proven by the Poissonian distribution of photocounts even during relaxation, which holds only if the field is ergodic. There is thus evidence that the biophoton field of a biological system is a fully coherent field. In turn, the hyperbolic oscillations around the delayed luminescence relaxation can be understood only in terms of couplings of coherent states. No non-living system is known that displays hyperbolic oscillations after light-induced re-emission.

However, as we now know, even squeezed states are possible. They may be squeezed in the position space $\langle q\rangle$ or in the momentum space $\langle p\rangle$, always satisfying the minimum uncertainty relation

$$\Delta p\, \Delta q = \frac{h}{2} \ . \tag{19.10}$$

In contrast to a coherent state, both Δp and Δq are variable in a squeezed state, whereby, keeping (19.10) valid, either $\Delta p \to 0$ and $\Delta q \to \infty$ or $\Delta q \to 0$ and $\Delta p \to \infty$.

Evidence has been shown by sub-Poissonian photocount statistics (for $\Delta p \to 0$, $r > 0$) in the case of an illuminated leaf and ultraweak photon emission from dinoflagellates [34]. The ordinary bioluminescence of all luminescent biological systems is triggered in general by biophotons which are, at least in the case of dinoflagellates, able to split into squeezed states with squeeze factors $r > 0$ ($\Delta p \to 0$) and $r < 0$ ($\Delta q \to 0$). This happens at the same time as the average photon number becomes smaller than 1. Further experimental work on squeezed states is still on the research program of the International Institute of Biophysics (IIB), Neuss.

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20 Quantum Theory of the Human Person

Henry P. Stapp

What is the future of quantum theory? Where will it go from here?

Quantum theory will go where it is most needed, which is into the effort to understand ourselves, and in particular the connection of our minds to our bodies.

Mind is the new frontier of science. The present century will be, for science, a century for advancing our understanding of the mind. Quantum theory is essential to progress here because classical physics leaves mind out: it relegates consciousness to the role of passive witness to a parade of physical events that are completely specified by local mechanical laws. Quantum theory, on the other hand, brings in the human participant in a subtle but essential way, and the quantum laws then suffice to explain the causal effects of our thoughts upon our actions.

The primary lesson taught by quantum theory is that the structure of empirically observed macroscopic phenomena cannot be understood within a conceptual framework in which the course of physical events is determined by local mechanical laws of the kind specified by the laws of classical physics. To deal with this incontrovertible fact the creators of quantum theory found themselves compelled to bring our minds into basic physical theory in a way less trivial than that of passive witnesses. This shift, though subtle in character, is sufficient to allow quantum theory to provide a causal explanation of the kinds of action of mind on brain that are now being observed in a growing number of neuro-psychological experiments conducted at major laboratories and universities. (For references, and a discussion of these experiments, see [10, 11].)

The nature of the causal connection between our minds and brains will, for many reasons, become increasingly a key focus of scientific research. The first reason is funding. Research goes primarily where the money is. The nature of the connection of our thoughts to our bodies is an important medical question. In an era of shrinking budgets, major funding will probably be channeled increasingly to research fields that are perceived to be vital to human needs. Already one sixth of the US GNP goes to the health industry, as contrasted to 2.8% for education and research. Thus there is a big pot from which to fund research into the nature of the conscious human organism.

But perhaps even more important than the question of our mental and physical health are cultural ramifications. We live immersed in a sea of ideas,

and the impact of these ideas is, today, at least as important to human destiny as our immediate physical conditions. This world of ideas is based directly upon our conception of what we ourselves are, and how we are connected to the forces that govern the universe.

The main contributors to the progress and development of these ideas are science, philosophy, and religion. The voice of science is strong in the construction of this intellectual milieu. But the message of science is blunted by the fact that it has hitherto been based mainly on classical physics, which proclaims us to be mechanical automata. That verdict conflicts with our deep intuitive idea of what we are, and leads to an impoverishment of values: value is reduced to the measure of mechanical well-being. Also, moral philosophy is undermined by the self-contradictoriness of the idea of striving to improve a future that was already mechanically determined before the earth was born. These difficulties with the classical-physics-based idea of man in nature tend to mute the voice of reason based on valid applicable science, and the more-than-just-matter conception of man that it provides.

The quest for an understanding of the mind–brain connection is rapidly becoming recognized as the most important and interesting task of science, at least on the biological side. Antonio Damasio begins his lead article in the recent special issue of Scientific American devoted to the mind–brain problem with the words [6]:

> At the start of the new millennium, it is apparent that one question towers above all others in the life sciences: How does the set of processes we call mind emerge from the activity of the organ we call brain?

The article by Francis Crick and Christoph Koch begins with the similar assessment [4]:

> The overwhelming question in neurobiology today is the relationship between the mind and the brain.

Some scientists [5] (p. 124) appear to believe that an adequate understanding of the mind–brain connection can be based essentially on classical physics, even though that theory is known to be unable to explain the observed macroscopic properties (e.g., electrical and thermal conductance, elasticity, etc.) of systems that depend sensitively – as conscious brains certainly do – on the behavior of their constituent atoms, molecules, electrons, and ions. This fatal failing of classical physical theory was remedied by the founders of quantum theory, who formulated their theory as a set of practical rules that specify how knowledge-seeking human agents should go about their tasks of first acquiring knowledge, and then representing that knowledge in a form that permits them to form expectations about the outcomes of their subsequent knowledge-seeking activities. The theory created by the founders is explicitly about connections between human experiences. It is a conceptual structure

that allows us to organize and make practical use of the knowledge we acquire. This profound shift of perspective is proclaimed by statements of the founders such as Heisenberg [7]:

> The conception of the objective reality of the elementary particles has thus evaporated not into the cloud of some obscure new reality concept, but into the transparent clarity of a mathematics that represents no longer the behavior of the particle but rather our knowledge of this behavior.

or Bohr [1] (p. 18):

> In our description of nature the purpose is not to disclose the real essence of phenomena but only to track down as far as possible the multifold aspects of our experience.

The subtle shift, alluded to above, in the role of the human agent is emphasized in statements such as:

> In the great drama of existence we ourselves are both actors and spectators. ([3], p. 15, [2], p. 81)

> The freedom of experimentation, presupposed in classical physics, is of course retained and corresponds to the free choice of experimental arrangement for which the mathematical structure of the quantum mechanical formalism offers the appropriate latitude. ([2], p. 73)

> To my mind there is no other alternative than to admit in this field of experience, we are dealing with individual phenomena and that our possibilities of handling the measuring instruments allow us to make a choice between the different complementary types of phenomena that we want to study. ([2], p. 51)

20.1 Von Neumann's Processes I and II

Quantum theory was rigorously formulated by John von Neumann [13] (p. 418), who identified two distinct processes, Process I and Process II.

Process II is the quantum analog of the classical process of motion and, like it, is governed by laws that are both local and deterministic. Process II is constructed from its classical counterpart by 'quantization', which replaces 'numerical values' by 'actions'. The effect of this change is to smear out the numerical values: it turns the physical state into a smeared out collection of overlapping classically conceivable possibilities. (This feature is not undone by the much-studied environmental decoherence effect, which effectively wipes out certain off-diagonal interference terms of the density matrix but does not restrict the evolution of the important diagonal elements.)

Process II is, however, not the whole story. It generates a continuum of overlapping physical possibilities that extend over a range of experientially realizable possibilities. Thus, for example, if Process II were the only process operating since the big bang then the physical structure representing the (center point of the) moon would extend over a large portion of the sky, contrary to the empirical facts. Some other process is needed to bring the physical state S of an observed system into conformity with human experience.

To tie the physically described state S to human experience the founders of quantum theory brought human agents into basic physical theory in a fundamental way. This was a radical move because the successes of the earlier classical theory were due in large measure to the policy of keeping human agents out. But in orthodox Copenhagen quantum theory the conscious actions of agents become crucial elements. In von Neumann's rigorous formulation of the theory, each such action is called a Process I intervention. This act is a preparatory act that amounts to posing a specific question with a Yes or No answer. (More complex cases can be built up by further decomposing the No possibility, but the essential point can be explained by focusing on a question with just two possible answers, Yes or No.)

A typical quantum question is: Will a Geiger counter set in some particular place be observed to fire at a time later than some specified time T_1 but earlier than some later specified time T_2? The preparatory action of putting this question to nature causes the state S to jump to a new state

$$S' = PSP + (1 - P)S(1 - P) \; .$$

The first term corresponds to the possibility that the feedback from nature will be a human experience of seeing the Geiger counter fire during the specified interval, and the second term corresponds to the failure of that specified experiential feedback to occur. These connections provide the critical correspondence between the mathematical/physical description and the psychological/experiential one.

20.2 The Action of Mind upon Brain

This Process I action by the agent involves a selection of a projection operator P from a continuum of alternative possibilities. This selection is, according to the orthodox rules of quantum theory, not specified by the physical state S of the system being examined or probed. This choice is taken to be a 'free choice' on the part of the experimenter/agent, as indicated by the statements by Bohr quoted earlier.

The reason why, in the original 'Copenhagen' version of quantum theory, the agent's choice must be regarded as 'free', in the specific sense that this choice is not determined by any known law of nature, is that in the Copenhagen formulation the experimenter stands outside the system that

is being probed, and his choice about how to conduct his probing action is considered to be up to him, not the system he is about to probe. In actual empirical practice, it is the prerogative of the observing agent, not the system being observed, to determine, from a continuum of possibilities, what kind of information or knowledge will be gathered by the agent during his act of observation. The system being observed does not make this choice.

Copenhagen quantum theory separates the dynamically unified physical world into two systems, the physical system S that is being examined, and the agent who is doing the examination. This latter system includes not only the stream of consciousness of the agent, and his body and brain, but also his measuring devices, which are regarded as extensions or parts of the agent. This extended agent is described in the sort of language that we use "to tell others what we have done and what we have learned" [2] (p. 3). This conceptual arrangement works well in practice, but it means that the theory cannot be viewed as a possible description of nature: it must be viewed as merely a set of rules for calculating expectations pertaining to relationships between human experiences.

Von Neumann's approach is to treat the entire physical world, including our own bodies and brains as belonging to the world described by the quantum laws. This tack circumvents the need to separate the dynamically unified physical world into two differently described subsystems. But it transfers those functions that the Copenhagen interpretation ascribes to the agent to what von Neumann calls the 'abstract ego' [13] (p. 421). This 'abstract ego' is what is left of the agent after his devices, body, and brain have been transferred to the physically described universe. This remainder, or residue, is the agent's stream of conscious experiences.

In von Neumann's formulation the stream of consciousness of the agent acts upon the brain of the agent. Thus in the Process 1 action

$$S \longrightarrow S' = PSP + (1 - P)S(1 - P) \ ,$$

the operator P acts non-trivially only on the brain of the agent. The choice of the operator P is ascribed to the abstract ego of the agent, namely his stream of consciousness. This provides, in principle, an opening within quantum dynamics for a possible action of a conscious mind upon the associated physically described brain.

20.3 Ideo-Motor Action

According to William James's ideo-motor theory of the connection between Volition and Action [8], willful action is associated with a prolongation of attention to the idea of the intended action. Accordingly, let it be assumed that each possible course of action that is conceivable to – and executable by – an agent is represented in the brain of that agent by an associated pattern

of brain activity that if held in place for a sufficiently long period of time will tend to cause that action to occur. I call this pattern a 'template for action'. It is the 'neural correlate' of the idea of that action. This pattern of brain activity is specified by a projection operator P that singles out from the morass of possible states of the brain of the agent those in which the associated template for action is activated. This projection operator P defines a possible Process 1 action. But the choices of which Process 1 interventions will occur are, according to quantum theory, 'free choices' made by the agents.

The key question then becomes: Can an agent's freedom to choose which Process I interventions occur have any effect on his bodily action?

The answer in von Neumann quantum theory is Yes! The agent's choices of Process 1 interventions can exert a huge effect on the physical activity of his brain, and this can have in turn a large effect on his bodily behavior. The simplest way to achieve such a result is via the quantum Zeno effect.

20.4 The Quantum Zeno Effect

Suppose S describes some slowly changing degrees of freedom of the brain. Suppose a sequence of 'freely chosen' Process I events consists of a rapid repetition of events with the same P. Then the laws of quantum theory entail that S will be trapped in the subspace of states of the form PSP if the original state has this form. Transitions to the other outcome $(I-P)S(I-P)$ of the Process 1 interventions are suppressed by virtue of the quadratic dependence upon the short time interval t between interventions of

$$(I-P)\exp(-\mathrm{i}Ht)(PSP)\exp(\mathrm{i}Ht)(I-P) = O(t^2)\ .$$

That is, suppose a Process 1 event, followed by Nature's choice of an outcome, lands you in the Yes state PSP. And suppose, your mental effort, within the range of 'free choices' available to you, can activate a rapid repetition of Process I events all associated with the same P. If the repetition rate is sufficiently fast then, due to the quadratic dependence on the time interval t between successive Process 1 events, there will be almost no transitions to the No states of the form $(I-P)S(I-P)$ over a long peiod of time, even if strong forces would tend to quickly move one out of the space specified by PSP in the absence of the rapid sequence of process 1 interventions.

One verifies the quadratic dependence on t by observing that a replacement of either one of the two exponentials by the zeroth order term 'unity' gives a null contribution to the transition probability, due to the constraint

$$P(1-P) = (1-P)P = 0\ ,$$

which is imposed by von Neumann's rules on the operators P.

The lowest order term is therefore of second order in t. This entails that if by means of his 'free choices' an agent can increase the rapidity of the Process

1 interventions to the point of activating the quantum Zeno effect, thereby holding in place some template for action, then the agent can, by exercising his 'free choice', influence his bodily actions.

This scenario is precisely in line with James's description of the effect of will:

> The essential achievement of the will, in short, when it is most 'voluntary', is to attend to a difficult object and hold it fast before the mind. [8]

> Everywhere, then, the function of effort is the same: to keep affirming and adopting the thought which, if left to itself would slip away. [8]

This explanation of the effect of mind on brain is strictly quantum mechanical. The classical-physics approximation eliminates the latitude provided by the uncertainty principle within which the play of free choice operates. Thus it closes the door opened by quantum theory to the possibility of a genuine causal influence of our thoughts, ideas, and feelings upon our physical actions.

William James clearly recognized the difficulty within classical physics of allowing mental effort to make a physical difference. He apparently understood what could then hardly be uttered by a man of science, namely that classical physics must be wrong. The prescient final words of his book [9] are:

> ... understand how great is the darkness in which we grope, and never forget that the natural-science assumptions with which we started are provisional and revisable things.

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21 Roundtable Discussion III: Information and Observation

Smolin: When I hear the word 'information', I think of Shannon. Information is then defined in a particular context, which is a channel, a sender and a receiver. When information is used in the context of 'is the world deterministic or not?', I get very confused. This overuse of the word 'information', to the point that somehow the laws of physics become information processing and so forth – I just don't understand. I think it's a misuse and I'd like to discuss it.

Rovelli: I feel challenged by that because I myself have been talking about information. The information I talked about is exactly the definition of Shannon, not the popular one, but the one in his book,[1] where you have two systems with many states, and there is a possible state of the couple but there is only a restricted subset of all joint states. Information is simply the way of counting the allowed joint possible states. The fact that there is a common allowed state tells you that if you know something about one system you also know something about the other one. This is exactly what you need in communication theory when you have a channel, a receiver and a transmitter. So I have two systems. If there is a quantum correlation of the two, I can say that, if this system is up, the other is down, and vice versa. This is what I mean by information, period.

The question is to Gerard ['t Hooft] about what *he* means by information. This morning you said that when there is a black hole there is information proportionate to the area. And then you said that this is very surprising because we are used to information being proportionate to the volume – and out of that comes a long story. I know from the geometry of the Earth that there are horizons, and that I can't get visual information from what's behind the horizons because of the geometry of the Earth. I know from general relativity that there are horizons like the horizon that bounds the black hole such that anybody outside can't get information about what's going on inside. In all these situations, particularly in GR, there is a precise distinction between information available to me and the total information including the one available on other side of the horizon. What is peculiar about black holes

[1] C.E. Shannon: *The Mathematical Theory of Communication* (University of Illinois Press, 1949).

is of course that nobody outside is going to get any information about what is inside. But if an observer goes inside she will have information about that. So there are two distinct notions of information. One is the information in a region of space, and the other one is information that I can get staying outside, from that region of space. At some point you ['t Hooft] said you were surprised that the information in a certain region is proportionate to the area, not the volume. But are you talking about the information available from the inside, or are you talking about the information available to anybody else? Because if you are talking about the information available from the outside, this does not imply that the number of degrees of freedom accessible from inside should be bounded by the area. If, on the other hand, you are talking about the information that anybody could have by going inside, I don't understand the consequences of what you said.

't Hooft: I think there are two questions here. One question, of principle, is the one that Lee [Smolin] also raised: *What is* information? The whole universe is only in one state, and will always be in one state. So you can say there is only one bit of information in the whole universe, and this is clearly wrong. The reason why it's wrong is that what we are doing when we are measuring is to consider all sorts of alternative outcomes which aren't realized by nature. So when we set up a measurement, or when we want to say something about a system, we first make a list of all the alternative possibilities, so the universe could be in a state that we are in some other town, or something like that. These are all alternatives, so we write down a complete list of alternatives, then we make a measurement and say that of all the alternatives this one or that is realized. So if you talk about information in the universe, you ask: What is a reasonable list of alternatives? And how big is that list? And with a black hole you ask: How big is the list of reasonable alternative states that the black hole can be in? The answer is: That list is the number which precisely fits on the horizon if you are allowed to write one bit per surface element.

By the way, that is the information which can be re-assessed by an outside observer. It's not information that disappeared behind the horizon, but information that comes back to him. And that's absolutely essential and important.

The best way to describe this would be to write down a model of the black hole, which is a lousy and stupid model, but it works as far as information goes. You build a brick wall outside the horizon, about one Planck length or a couple of Planck lengths outside the horizon. Then you give the black hole an atmosphere, such that the particles which went all the way outside the black hole are just Hawking radiation. It turns out that if you place the brick wall a couple of Planck lengths outside the horizon, then the atmosphere of the black hole, these Hawking particles, precisely represent that partial information. Then you can ask: How many different ways can this atmosphere

of the black hole look? Where can all these particles be, given the constraints of quantum mechanics? And you find that the black hole can be in a number of states. These quantum states can be counted. It is the same number of alternative states I mentioned earlier, that you can count.

Rovelli: Surely the number of possible states inside may very well be proportionate to the volume, not to the area?

't Hooft: Yes, but those states are irretrievable for the outside observer. I'm talking about everything you can see about the black hole from the outside. What's inside cannot be recovered.

Rovelli: But what about your holographic hypothesis?

't Hooft: The whole idea is as follows. One would have thought – now, this may be wrong – that the properties of the black hole horizon can be retrieved by doing a general coordinate *transformation*, saying that the horizon is just a region of regular spacetime. All you have to do is the Rindler space transformation, or something like that, near the horizon, and then you can make a mapping onto the physics of ordinary flat space. So, if you want to know what the black hole does, all you have to do is physics of flat space which we thought we understood. This is a big mistake. We don't understand the physics of flat space at all! This is what comes out of this argument: that we don't actually know the physics of flat space! It's because we haven't understood what happens at Planckian distance regimes. And that's what the answer to this exercise is. But I thought we should have understood what happens in flat space, and then, if you know that, you can deduce the properties of the black hole from the properties of flat space. If that were so, the total number of alternative states a black hole could be in should be equal to the total number of alternative states that you have in flat space – that is, if you believe that there is a one-to-one mapping. Well, somewhere there is a mistake in all this, because this mapping doesn't work, since you would think that in flat space the total number of alternative states is proportionate to the volume. But it's not. If it were a black hole, it would be just the surface. So somewhere along the line a mistake has been made. It's very important to ask where the mistake was made – how come, how can we cure it, and how can we get a superior theory?

Butterfield: When counting states, I would at first think that one is counting the dimensionality of a Hilbert space, since any one of these spaces has continuously many states. But then I'm worried when you say that the number would be expected to scale with the volume, since we know that $L^2(\mathbb{R}^2)$ or $L^2(\mathbb{R}^3)$ are both of denumerable dimension. So what exactly is the count-

ing here?

't Hooft: The counting goes by indirect argument, and that is the strange thing about the whole situation. I wish I had a direct answer to the question, because you are perfectly right: $L^2(\mathbb{R}^2)$ has an infinite number of states, so how come it's strictly finite? So the only possible answer is that somehow there is a cutoff. I am not working with $L^2(\mathbb{R}^2)$, I'm working with functions defined on a lattice of spacetime, something like that. And on top of that there is some kind of an exclusion principle that says that no two bits are allowed to sit on the same spot. So here we have the exclusion principle. You can only work with a fermionic theory on a lattice. It must be like this because I got this answer of one bit per Planckian surface element.

Smolin: I have another question about the holographic principle. There are two views which you might call the weak version and the strong version of the holographic principle, and I'm curious which one is yours.

The weak version is: Given a spacelike surface somewhere in spacetime – I'm taking on board what you said that they should not be just at the horizon of a black hole but anywhere in spacetime – there is a limitation to the amount of information that might be traversing that surface from its causal past to its causal future, which is given by one bit of information per every four Planck areas.

The strong version is: If I have a closed surface of arbitrary size, there is a limit to the amount of information that might ever be measured even by someone going into the interior of the surface, which is proportionate to the area of the surface.

I am trying to distinguish between the two because one can construct in the cosmological context counterexamples to the second version. For example, in an inflationary universe, you can have a region that grows up and inflates and then re-heats and has an entropy on the interior much larger than the bound that is proposed.

't Hooft: That has to do with my previous remark to Carlo [Rovelli], that you have to make a list of alternatives and you can make the list as long as you want. But most of these alternatives are fundamentally impossible. So you can just wipe them out. There is some freedom there. That is important in the definition of information which also comes in from all sorts of definitions of entropy in thermodynamics and things like that. If you ask: What is entropy?, you first have to say: "I'm looking at a sort of substance", for example, "I want to know the entropy of the water in this glass", but you have to ask what is the total number of states that the water can be in. And you have to ask what's the total number of states that water can take such that it still looks like water. I don't want it to look like alcohol or gasoline or whatever, because that is clearly something I can distinguish. It's not that. But if I say

I'm counting all states which this could be in, without making any change in experience or any change in its coarse-grained physical properties, then that is the entropy. So I think that if you use that definition for a gross surface, by saying: "What is the reasonable upper limit on the total number of states that you can choose from to describe the inside?", then you get the Beckenstein bound.

In cosmology, you sometimes talk about the surfaces that could be either the inside or the outside. And of course, outside surfaces have far more states than inside surfaces. And if you have a closed universe, you don't know what's inside and what's outside, so you get into this mystery, when you have to ask: On which side of the surface are the states that I am counting? I do not have the answer, but one could imagine the following. The inside observer has only a limited memory, so he can only imagine a limited number of states for the outside observer, for the outside world. Even if the outside world is richer than the inside observer can imagine. And then, this total list of alternatives is limited for that reason.

I say this because I suspect that the data at the surface reflect the expression $\langle \text{out}|\text{in}\rangle$, where the bras $\langle \text{out}|$ stand for the states outside, and the kets $|\text{in}\rangle$ for the states inside, and this matrix must be unitary.

Saunders: I'd like to ask Carlo [Rovelli]: Suppose that relative to Carlo some experimental outcome is 'red' while, relative to Anton [Zeilinger], Carlo and the apparatus are in superposition at the same time. So both these two statements are made at the same time. I'd like to understand how to reconcile these two statements with one another.

Rovelli: My main suggestion is to forbid ourselves to use the point of view of God. Do not compare two different observers, unless you are, for instance, a third observer who interacts with the two. In order to make this comparison you have to have a quantum mechanical interaction. So, very simply, the answer is like that of special relativity: I am telling you that, with respect to *this* observer, *this* comes first and *this* comes second, whereas, with respect to *another* observer, the order is the opposite. Intuitively one might think that this cannot be. But really there is no contradiction.

Saunders: Let me push this a little bit further. If Anton is correct in attributing the superposition to this combined system for himself, it seems to me that it is also the case that if Anton subsequently opens the box and makes a measurement he should find on some occasions that when he opens the box he finds not 'Carlo seeing red' but 'Carlo seeing blue'.

Rovelli: Yes, for himself!

Saunders: That's right. Although relative to Carlo he is seeing red.

Rovelli: Not 'although' – he is seeing red, period! Now if you want to compare the two, you need some quantum communication for comparing the two.

Saunders: I'm trying to understand how: Anton has a Carlo who is seeing blue, whereas Carlo himself sees red?!

Rovelli: In the same sense in which two things can be different with respect to two different reference frames. These are not facts of Nature, 'seeing red' or 'seeing blue'. 'Seeing red' is like 'being red' – it's a statement about a physical system, which is only true when referred to some observer.

Saunders: I have been careful to refer it to observers. Relative to Anton there is a Carlo seeing blue, but relative to Carlo there is a red outcome. I still don't know how to understand this unless there are two different Carlos!

Rovelli: What I'm saying is that quantum mechanics is not about a factual state of affairs of the world.

Smolin: The dynamics, and/or the principle that Carlo calls the von Neumann principle of consistency,[2] must ensure that there will never be a case where Carlo will see an eigenstate of red for his experiment and Anton will see an eigenstate which is orthogonal to red.

Saunders: When Carlo sees red here at 12 o'clock, Anton, from the outside, correctly attributes to him at 12 o'clock a state of superposition. You are saying that there will never be a case that when Anton opens the box he will find Carlo seeing blue?

Smolin: Yes.

Saunders: In that case I don't understand how this attribution of a superposition can be correct.

Rovelli: What I'm saying is that what we cannot speak about we must pass over in silence: if you take one observer and you follow all the information that she has, all the statements that she can verify, for all the time you want and with all the interactions you want, you get complete consistency. If you don't forget about that, and you take another observer, and you follow all the information that she has, and all the statements that she can make, you get complete consistency. Furthermore, I'm saying that if the two observers

[2] C. Rovelli: Relational quantum mechanics, Intl. J. of Theoret. Phys. **35**, 1637 (1996).

communicate, this doesn't disturb in any way the consistency of one or the other: the rest is silence.

Unidentified: Time for dinner!

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