EPR et l'inégalité de Bell vus par un probabiliste
EPR and Bell's inequality from a probabilist's perspective

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Discuss the Einstein-Podolski-Rosen "paradox", Bell's inequalities, the experiments by Aspect and Co, and their implication on the rôle of probability in the description of reality.

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- A probabilist's view on Bell's inequalities
- "Proofs" that our world is not "classical"
- Why we need "quantum probability"

Claim

No classical (= realistic, non-contextual, local) theory can describe can correctly describe our world, if quantum mechanics is correct (as all experiments have confirmed so far).

Quantum probability spaces

Definition

A quantum probability space is a pair (A, φ) consisting of a (Von Neumann) algebra A and a (normal) state $\varphi : A \to \mathbb{C}$.

Example (Classical Probability)

A classical probability space is a triple $(\Omega, \mathcal{F}, P)$ where

- Ω is a set, the sample space, the set of all possible outcomes.
- $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is the set of events.
- $P : \mathcal{F} \to [0, 1]$ assign to each event it's probability.

This description/model of random events is consistent with the idea that randomness is due to a lack of information.
If we knew which $\omega \in \Omega$ is realized, then the randomness disappears.

Example (Classical Probability, cont'd)

To a classical probability space $(\Omega, \mathcal{F}, P)$ we can associate a quantum probability space (A, φ), take

- $A = L^\infty(\Omega, \mathcal{F}, P)$, the algebra of bounded measurable fonctions $f : \Omega \to \mathbb{C}$, called the algebra of random variables or observables.
- $\varphi : A \ni f \mapsto E(f) = \int_\Omega f \mathrm{d}P$, which assigns to each random variable/observable its expected value.

$(\Omega, \mathcal{F}, P)$ and (A, P) are essentially equivalent (by the spectral theorem).

Quantum probability spaces

Exemple (Quantum mechanics)

Let H be a Hilbert space, with a unit vector ψ (or a density matrix ρ). Then the quantum probability space associated to (H, ψ) (or (H, ρ) is given by

- $A = B(H)$, the algebra bounded linear operators $X : H \to H$. Self-adjoint (or normal) operators can be considered as quantum random variables or observables.
- $\varphi : B(H) \ni X \mapsto \varphi(X) = \langle \psi, X\psi \rangle$ (or $\varphi(X) = \mathrm{tr}(\rho X)$).

Question

Can the randomness still be explained by a lack of information?

One q-bit

Example: spin-$\frac{1}{2}$ or polarisation of a photon

$H = \mathbb{C}^2$. The most general state vector is of the form

$$\psi = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}.$$

with $\theta \in [0, \pi)$, $\phi \in [0, 2\pi)$, $|0\rangle = |\uparrow\rangle$, $|1\rangle = |\downarrow\rangle$, and can be visualized as the point (θ, ϕ) on the unit sphere (Bloch sphere) in $\mathbb{R}^3$, i.e. the vector

$$\begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix}.$$

One q-bit

Example: spin-$\frac{1}{2}$ or polarisation of a photon, cont'd

Density matrices are of the form

$$\rho(x,y,z) = \frac{I + x\sigma_x + y\sigma_y + z\sigma_z}{2}$$

with $x, y, z \in \mathbb{R}$, $x^2 + y^2 + z^2 \leq 1$, where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that

$$|\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & 1-\cos\theta \end{pmatrix} = \rho\begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix}.$$

Example: spin-$\frac{1}{2}$ or polarisation of a photon, cont'd

Observables (self-adjoint operators) are of the form

$$X = a|\psi\rangle\langle\psi| + b|\psi_\perp\rangle\langle\psi_\perp|,$$

for $a, b \in \mathbb{R}$, ψ a unit vector, $\psi_\perp$ orthogonal to ψ (unique up to a phase).
In an experiment, X takes values a and b, with probabilities

$$P(X = a) = \varphi\big(|\psi\rangle\langle\psi|\big) \text{ and } P(X = b) = \varphi\big(|\psi_\perp\rangle\langle\psi_\perp|\big)$$

E.g., for $\phi = \langle\psi', \cdot\psi'\rangle$ the vector state associated to $\psi' = \cos\frac{\theta'}{2}|0\rangle + e^{i\phi'}\sin\frac{\theta'}{2}|1\rangle$, we get

$$P(X = a) = \big|\langle\psi, \psi'\rangle\big|^2 = \frac{1 + \cos\vartheta}{2} \text{ and } P(X = b) = \frac{1 - \cos\vartheta}{2}$$

where ϑ is the angle between ψ and ψ' on the Bloch sphere.